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MODEL PREDICTIVE CONTROL OF SYSTEMS WITH CHANGING SETPOINTS

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MODEL PREDICTIVE CONTROL OF SYSTEMS WITH CHANGING SETPOINTS

Thesis presented to Universidad de Sevilla
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Doctor of Philosophy.

POR

ANTONIO FERRAMOSCA

Seville, June 2011.

To Cecilia
To my parents and my sister

Acknowledgments

This thesis is the result of five years of studies and work at the Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla.

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Resumen de la Tesis presentada en la Universidad de Sevilla como uno de los requisitos necesarios para la obtención del grado de Doctor of Philosophy.

CONTROL PREDICTIVO DE SISTEMAS CON PUNTO DE OPERACIÓN CAMBIANTE

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Esta tesis trata el problema del diseño de un controlador predictivo para sistemas caracterizados por cambios en el punto de operación. La clásica formulación del controlador predictivo, para regular el sistema al nuevo punto de operación deseado, garantiza seguimiento de referencia en caso de sistemas que no estén sujeto a restricciones, pero no resuelve el problema cuando hay restricciones. En esos casos, un cambio de referencia puede producir una perdida de la factibilidad del problema de optimización por una de las siguientes causas: (i) la restricción terminal para el nuevo punto de equilibrio puede no ser un invariante y (ii) la region terminal para el nuevo punto de operación podría no ser alcanzable en N pasos. Para recuperar la factibilidad, se requeriría el recálculo del horizonte por lo que un cambio de referencia conllevaría el rediseño on-line del controlador, lo que no será siempre posible.
En este trabajo de tesis se presenta una nueva formulación de control predictivo que permite solucionar este problema. Las principales características de esta nueva formulación son: un punto de equilibrio artificial considerado como variable de decisión,

un coste que penalice la distancia entre la trayectoria predicha y el punto de equilibrio artificial, un coste adicional que penalice la distancia entre el punto de equilibrio artificial y el punto de equilibrio deseado, llamado *coste de offset*, y una restricción terminal extendida, el conjunto invariante para seguimiento. Este controlador garantiza estabilidad y factibilidad recursiva para cualquier cambio de referencia. En esta tesis se demuestra que una adecuada elección del coste de offset garantiza la propiedad de la optimalidad local del controlador. Además, se presenta una caracterización de las regiones en las cuales esta propiedad se cumple.
El coste de offset juega el papel de un optimizador en tiempo real (RTO) incorporado en el mismo controlador predictivo. Así, este coste de offset permite trabajar con plantas no cuadradas, o con puntos de operación no alcanzables. En este ultimo caso, el controlador lleva el sistema al punto de equilibrio más cercano, en el sentido que se minimiza el coste de offset. Además se demuestra que este coste de offset se puede formular como distancia a un conjunto. Esta formulación hace el controlador predictivo para tracking propuesto, adecuado también para problemas de control por zonas. En estos problemas el objetivo no es un punto fijo; es más bien una región dentro de la cual se desea que las salidas permanezcan. Para este caso, en la tesis se propone un controlador robusto basado en predicciones nominales y en restricciones contractivas.
En este trabajo se trata también el tema del control de sistemas de gran escala. Estos sistemas se pueden ver como una serie de unidades operativas, interconectadas entre ellas. Por lo tanto, esas plantas se pueden dividir en diferentes subsistemas que comunican entre ellos por medio de redes de varias naturalezas. El control total de esas plantas usando controladores centralizados - un solo agente controlando todos los subsistemas - es difícil de realizarse, por un lado por la elevada carga computacional, y por el otro lado por la difícil organización y el mantenimiento del controlador centralizado. Por lo tanto, una estrategia de control alternativa es el control distribuido. Se trata de una estrategia basada en diferentes agentes controlando los diferentes subsistemas, que pueden o no intercambiar informaciones entre ellos. La diferencia entre las diferentes estrategias de control predictivo, es la manera de tratar el intercambio de informaciones. En el control distribuido noncooperativo, cada agente toma decisiones sobre su propio subsistemas considerando solo localmente las informaciones de los otros subsistemas. Las prestaciones de la planta suelen converger a un equilibrio de Nash. Los controladores distribuidos cooperativo, por otro lado, consideran el efecto de todas las acciones de control sobre todos los subsistemas de toda la red. Cada agente optimiza un coste global, como por ejemplo un coste centralizado. Por lo tanto, las prestaciones de estos controladores convergen a un equilibrio de Pareto, como en el caso centralizado. En este trabajo de tesis se propone una estrategia de control predictivo para seguimiento distribuido cooperativo y se demuestra que el controlador lleva el sistema al óptimo del centralizado.

La tesis toma en consideración también los sistemas nolineales. En particular, el controlador propuesto se extiende al caso de sistemas no lineales y se proponen tres formulaciones, respectivamente con restricción terminal de igualdad, restricción terminal de desigualdad y sin restricción terminal. En particular, para la formulación con restricción de igualdad, se propone un método basado en el modelado LTV de las plantas. La idea es diseãr un conjunto de controladores locales, cuya región de factibilidad cubra el entero conjunto de puntos de equilibrio.
Finalmente, el trabajo de tesis trata el problema del diseño de controladores predictivos con optimalidad económica. Esta formulación considera un funcional de coste basado en objetivos económicos, en lugar del clásico funcional basado en errores de seguimiento, y provee mejores prestaciones con respeto al objetivo que los estándar controladores para seguimiento. En la tesis se presenta un controlador predictivo económico para objetivos econónmicos cambiantes. Ese controlador es una formulación híbrida entre el control predictivo para seguimiento y el controlador predictivo económico, dado que hereda la factibilidad garantizada para cualquier cambio del objetivo del primero, y la optimalidad con respeto al objetivo del segundo.

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This thesis deals with the problem of designing a model predictive controller (MPC) for process characterized by changes in their setpoint. The traditional MPC formulation as a regulation problem guarantees the setpoint tracking when there are no constraints but may not solve the problem when there are constraints on the plant. In this case, the change of setpoint may cause a loss of feasibility of the optimization problem, mainly because of two reasons: (i) the terminal set shifted to the new operating point may not be an admissible invariant set and (ii) the terminal region at the new setpoint could be unreachable in N steps. In this case, a re-calculation of an appropriate value of the prediction horizon is necessary to ensure feasibility, and this would require an on-line re-design of the controller for each set point, which can be computationally unaffordable.

In this thesis, a MPC formulation able to overcome this problem is presented. This formulation is characterized by the use of an artificial steady state considered as decision variable, the use of a cost function which measures the distance of the predicted

trajectory to the artificial steady state, an additional cost that penalizes the distance of the artificial steady state to the desired output (the offset cost function), and an extended terminal constraint, the invariant set for tracking. The thesis proves that a suitable choice of the offset cost function ensures the local optimality property of the controller. Moreover, the thesis presents a characterization of the region in which this property is ensured.

The offset cost function plays the role of a real-time optimizer (RTO) built in the MPC controller. This offset cost function can deal with non-square plant, and with unreachable setpoints. In this case the system is driven to the closest (in the sense the offset cost is minimized) admissible steady state. It is also proved that this function can be formulated as a distance to a set. This formulation makes the MPC for tracking suitable also for zone control problems, where the desired setpoint is not a fixed-point, but the output are desired to lie in a set. For this case, a robust MPC for tracking formulation is also presented. This MPC is based on the calculation of nominal prediction and on the use of restricted constraints.

The thesis also addresses the control problem of large scale systems consisting of many linked units. These systems can be considered as a number of subsystems connected by networks of different nature. The overall control of these plants by means of a centralized controller - a single agent controlling all subsystems - is difficult to realize, because of the high computational burden and the difficulty of managing the interchanges of information between the single units. Hence, distributed control is an alternative control strategy; that is, a control strategy based on different agents - instead of a centralized controller - controlling each subsystems, which may or may not share information. The difference between these distributed control strategies is in the use of this open-loop information: noncooperative controllers, where each agent makes decisions on the single subsystem considering the other subsystems information only locally and which make the plant converge to a Nash equilibrium; cooperative distributed controllers consider the effect of the control actions on all subsystems in the network and makes the system converge to the Pareto optimum. A cooperative distributed MPC for tracking linear systems is presented in this thesis. The proposed MPC is able to guarantee recursive feasibility and convergence to the centralized target.

The thesis also deals with nonlinear systems. In particular, the MPC for tracking is extended to deal with nonlinear systems and formulated with equality terminal constraint, inequality terminal constraint and without terminal constraint. The calculation of the terminal ingredients, in the case of inequality terminal constraint, is not trivial. This thesis proposes a method for their calculation, based on an LDI. The idea is to design a set of local predictive controllers, whose feasible regions cover the entire steady state manifold.

Finally, the thesis focuses on the topic of economic MPC. This MPC formulation is characterized by considering an economic function as stage cost, providing better optimal performance with respect to the economic criterion. The thesis presents an economic MPC for changing economic criteria, which is able to provide the optimality properties of the economic MPC, and at the same time to ensure feasibility under any change of the economic criterion, enlarging also the domain of attraction of the controller.

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Notation

x'	Given a vector x, x' denotes its transposed
T'	Given T, T' denotes its transposed
$T > 0$	Given a symmetric matrix T, $T > 0$ denotes that it is positive definite
$T > P$	$T - P > 0$
$T_\perp$	Denotes a matrix such that $T'T_\perp = 0$ and the matrix $[T, T_\perp]$ is a non-singular square matrix.
$\|x\|_P$	The weighted Euclidean norm of x, i.e.,$\|x\|_P = \sqrt{x'Px}$, being $P > 0$ a symmetric matrix
$\|A\|_p$	Denotes the p-norm of the matrix A
$\mathbf{x}$	Is a generic vector defined as $\mathbf{x} \triangleq \{x(0), x(1), \dots\}$
(a, b)	$[a', b']'$
$\mathbf{1}$	Denotes a vector of appropriate dimension where each component is 1
I_p	Denotes the identity matrix of order p
$\mathbf{0}_{n,m}$	Denotes a matrix of zeros $\in \mathbb{R}^{n \times m}$.
$\mathbb{N}$	Set of natural numbers
$\mathbb{I}$	Set of integer numbers
$\mathbb{I}_{l:r}$	Given two integers, $l \leq r$, the set $\mathbb{I}_{l:r} = \{l, l+1, ..., r-1, r\}$
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_{\geq 0}$	Set of positive real numbers
$\mathbb{R}^n$	Space of n-vector of real values
$\mathbb{R}^{n \times m}$	Space of $n \times m$ matrix of real values
$\mathcal{B}$	The unitary ball $\mathcal{B} = \{b \in \mathbb{R} : \|b\|_\infty \leq 1\}$

$\mathcal{D}$ is a C set	If $\mathcal{D}$ is compact, convex and non empty
$vert(\Gamma)$	Is the set of vertexes of a given C set Γ
$int(\Gamma)$	Denotes the interior of a set Γ
$Proj_a(\Gamma)$	Consider $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$, and set $\Gamma \subset \mathbb{R}^{n_a+n_b}$, then the projection operation is defined as the following set $Proj_a(\Gamma) = \{a \in \mathbb{R}^{n_a} : \exists b \in \mathbb{R}^{n_b},\, (a,b) \in \Gamma\}$
$\mathcal{U} \oplus \mathcal{V}$	Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v : u \in \mathcal{U}, v \in \mathcal{V}\}$
$\mathcal{U} \ominus \mathcal{V}$	Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Pontryagin set difference is: $\mathcal{U} \ominus \mathcal{V} \triangleq \{u : u \oplus \mathcal{V} \subseteq \mathcal{U}\}$
$M\mathcal{V} \subset \mathbb{R}^n$	For a given matrix $M \in \mathbb{R}^{n\times m}$ and a set $\mathcal{V} \subset \mathbb{R}^m$, the set $M\mathcal{V} \subset \mathbb{R}^n$ denotes the set $\{y = Mv,\, v \in \mathcal{V}\}$
$\lambda\mathcal{X}$	For a given positive λ, $\lambda\mathcal{X} = \{\lambda x : x \in \mathcal{X}\}$
$\bigoplus_{i=0}^{n} \mathcal{W}_i$	Denotes the Minkowski addition of the n sets $\mathcal{W}_i$ where $i = 0, \ldots, n$ $\bigoplus_{i=0}^{n} \mathcal{W}_i \triangleq \mathcal{W}_1 \oplus, \ldots, \oplus \mathcal{W}_n$
Ω_t^a	Denotes the invariant set for tracking, which is defined in the augmented state (x, θ)
Ω_t	Denotes the projection of Ω_t^a onto x
inf	Infimum or greatest lower bound
min	Minimum
sup	Supremum or least upper bound
max	Maximum
arg	Argument or solution of an optimization

Acronyms

CSTR	Continuous Stirred Tank Reactor
DMC	Dynamic Matrix Control
E-MPC	Economic Model Predictive Control
E-MPCT	Model Predictive Control for a Changing Economic Criterion
FCC	Fluid Catalytic Cracking
ISS	Input-to-State Stability
KKT	Karush-Kuhn-Tucker Optimality Conditions
LDI	Linear Differential Inclusion
LMI	Linear Matrix Inequality
LP	Linear Programming
LQR	Linear Quadratic Regulator
LTI	Linear Time Invariant
LTV	Linear Time Varying
MPC	Model Predictive Control
MPCT	Model Predictive Control for Tracking
mp-QP	Multiparametric Quadratic Programming
OPC	Ole for Process Control
PLC	Programmable Logic Controller
QP	Quadratic Programming
RPI	Robust Positively Invariant
RTO	Real-Time Optimizer

Chapter 1

Introduction

The aim of this chapter is to describe the motivation and objectives of this thesis and to introduce the research work done. First of all, the problem of controlling systems characterized by changes of their operating points is introduced as motivation of this thesis. Then, an overview of the control strategy used in this work, Model Predictive Control (MPC), is given. Finally, the issues for which this work proposed a solution, will be presented.

1.1 Motivation of the thesis

In recent years, operation techniques in the process industries has made important progress, due to the need for production in a safe, clean, and competitive way and satisfying the necessities of the market, with respect to both demand and quality. Two reasons justify this fact: on one hand, the need to satisfy the necessities of a market which is even more diversified because of its social and cultural habits and the need for strict safety controls on products as well as variety and quality, which all results in a shorter product life cycle. On the other hand, the need to favor sustainable growth, minimizing both environmental impact and resource consumption. Both factors contribute to the desire for the most efficient production which satisfies requirements and limits imposed on the products. For all this, it is desirable to look for control techniques which provide control laws that optimize some efficiency criteria and guarantee the satisfaction of the limits imposed on the products. Model predictive control is one of the few techniques which permit this problem to be solved (Camacho and Bordons, 2004).

Typically, in industries, the processes are operated at optimal operation points at which they should remain in order to maximize their efficiency. However, we cannot just talk of just one optimal operating point: processes are in fact characterized by a range of operating points at which they should be for a time. The selection of a point from this range depends on the variety of product, economic request or situations in which the plant might be.

The aim of this work is to develop an advanced control strategy for constrained processes with changing operating points, that permits efficient, flexible and integral operation in such a way that, using the available resources, the security and quality of products are guaranteed.

1.2 Control of plants with changing operating points

The proposed control problem, is characterized by two main aspects, which define the entire nature of this problem. First of all, the large range of operation of the plant are considered, and this stresses the nonlinear nature of its dynamics (implicit in the equations associated to mass, energy and momentum balances) and the uncertainty level (structural and parametric) associated to its state space representation. Moreover, it is important to remark that this kind of plant are characterized by complex dynamics, usually defined by systems characterized by coupled algebraic, ordinary differential or partial differential equations.

The second determining aspect is the presence of constraints. These constraints can be limits on the manipulated variables variables, or limits on the process variables. They can derive from the physical limits of the variables or from limits in the plant evolution zone due to economical, environmental or operational reasons. The presence of constraints influences systems behavior, in particular stressing their nonlinear nature. They can also cause performance loss and instability (Mayne, 2001).

The traditional way to approach this control problem consists on a multilayer control structure (Tatjewski, 2008). This structure is in general a hierarchical structure where the lower level control deals with the regulation of the plant. This task is usually done by PID or by programmable logic controllers (PLC) connected in a network. The higher level control is usually a multivariable advanced controller, which determines the input of the lower level control in order to keep the system at the desired operating point. This operating point is calculated by a setpoint optimizer - a real time optimizer (RTO) - according to data, economic criteria, or information coming from the plant. This structure is shown in figure 1.1. The task of the high-level controller is to keep

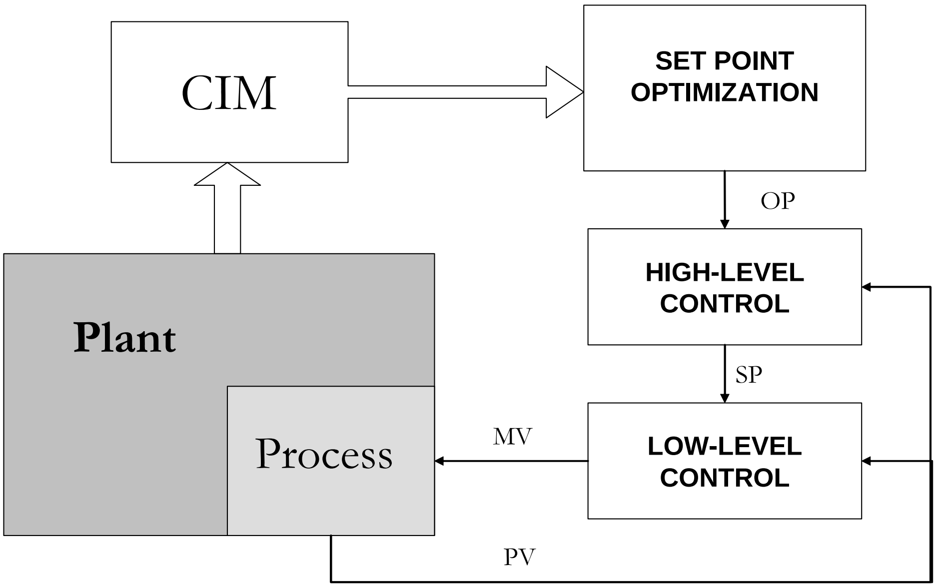


Figure 1.1: Hierarchical control structure.

the system at the desired operating point. When the RTO provides a change of the setpoint, the high-level control has to react in order to move the system to the new operating point. This task is not trivial, due to the changes of dynamics that may appear at the new setpoint, and to the necessity of guarantee constraints satisfaction. In order to manage significant changes of the operating point, the high-level controller is usually designes as a two-layer structure (Becerra et al., 1998): the lower layer deals with the regulation of the system, while the upper one has the task of adapting the controller to the new setpoint, that is managing the transitory when the operating point changes. This scheme is shown in figure 1.2. Adaptive controllers such as the classic gain scheduling belong to this scheme. Another kind of adaptive advanced controllers are the so called *reference governors* (Gilbert et al., 1994, 1999). The aim of the reference governors is to manage, in a certain way, the references, in order to avoid constraints violation when the setpoint changes. The design of this kind of controller does not take into account the efficiency or the performance of the process. The only aim is to avoid the violation of the limits. This aspect is discussed in (Bemporad et al., 1997). Reference governors are also successfully used in case of nonlinear systems (Bemporad, 1998b; Angeli and Mosca, 1999; Gilbert and Kolmanovsky, 2002).

Model predictive control (Qin and Badgwell, 1997) is one of the most successful advanced control strategies, due to the nature of its control law, based on the mini-

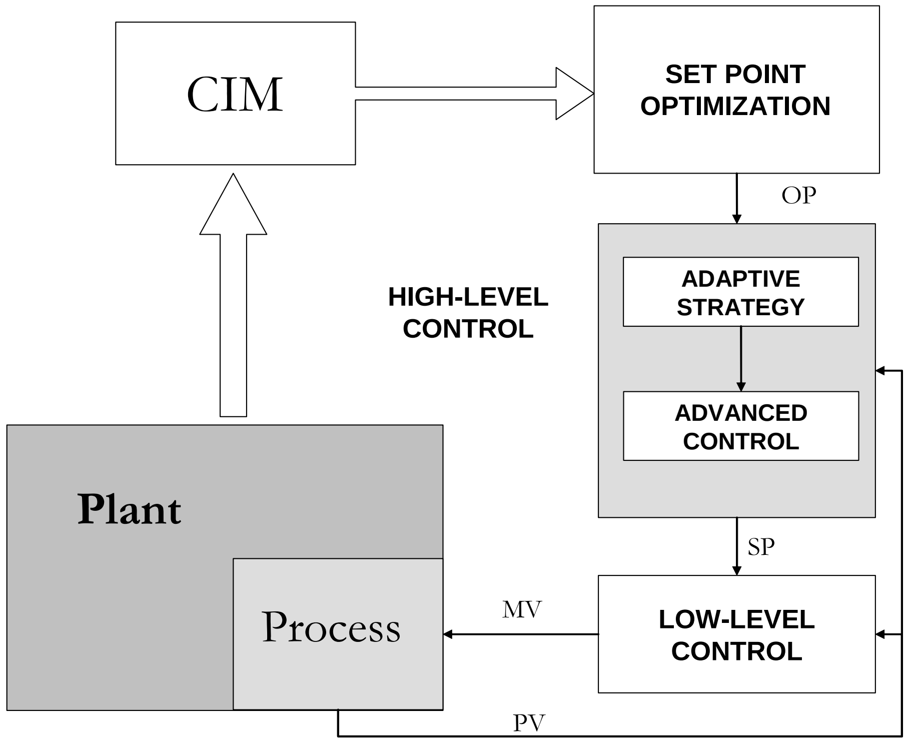


Figure 1.2: Hierarchical control structure with adaptive higher level.

mization of a constrained optimum criterion. In the case of model predictive control, there are a lot of formulations oriented to the management of large transitions. These controllers are able to calculate the optimal control action on the basis of a performance criterion, allowing significant changes of the operating point. The stability guarantee is based on a hierarchical structure, like the one shown in figure 1.2: the higher sublevel deals with the commutation between the predictive controller and the other controller oriented at the recuperation of the system in case of loss of feasibility.

Integral control is another way of approaching the problem. In this case, the advanced control strategy (generally predictive control) is associated to an economical objective. Hence, some of the optimization tasks move from the setpoint optimizer to the advanced controller, in order to incorporate the cost associated to the transitions into the operation point determination. In (Becerra and Roberts, 1996; Becerra et al., 1997, 1998) a different way to integrate model predictive control with on-line optimization of economical objectives are considered, such as the multi-objective control problems in which both the regulation objective and the economical one are minimized. In (Vesely et al., 1998) a method for complex system steady state optimization that can be solved by means of algebraic equations, is presented. However, these works do not provide stability, robustness and convergence studies.

In conclusion, the hierarchical structure guarantees stability and constraint satisfaction but gives worse performance than integral control due to the independent design of the control layers. The integral control structures do not provide stability and constraint satisfaction. Hence, in case of operating points large transition it is necessary the design of a control strategy which permit the unification of the integral control at only one level, minimizing a performance index and guaranteeing at the same time constraints satisfaction and stability.

Model predictive control (MPC) is one of the most successful techniques of advanced control in the process industry (Camacho and Bordons, 2004), because it allows the control of systems subjected to constraints, minimizing an optimum criterion and guaranteeing stability and convergence to the equilibrium point (Mayne, 2001; Rawlings and Mayne, 2009). In figure 1.3 is shown how the predictive controller replaces the two-layer control structure. Hence, in this thesis, it is proposed as the strategy to approach the considered problem.

Model predictive control has been one of the hot topic in the academy in the last years, and now the control problem and the theoretical framework are well understood (Mayne et al., 2000; Rawlings and Mayne, 2009; Limon, 2002). Moreover, it has been proved that MPC is also a powerfull control strategies for the case of robust control problems with constraints (Mayne et al., 2000; Limon, 2002; De Nicolao et al., 1996; Magni et al., 2001c; Fontes and Magni, 2003; Limon et al., 2005, 2006a). The stabilizing design of MPC controllers is based on the calculation of invariant sets (Blanchini, 1999; Bertsekas, 1972).

In the case of linear system with or without uncertainties, there exist efficient control strategies which allows to control the plant ensuring stability and constraints satisfaction. Some techniques in order to simplify the optimization problem allowing its online solution have been proposed in (Alamo et al., 2005). At the same time, techniques devoted to the calculation of the explicit solution of the MPC have been presented in (Bemporad et al., 2002; Jones et al., 2007; Zeilinger et al., 2008; Jones and Morari, 2010).
As for nonlinear systems, the control problem is more complex and requires the solution of a nonlinear optimization problem. In order to relax the computational burden, conditions for ensuring stability even in case of suboptimal solutions of the optimization problem, have been proposed in (Scokaert et al., 1999). Robust nonlinear model

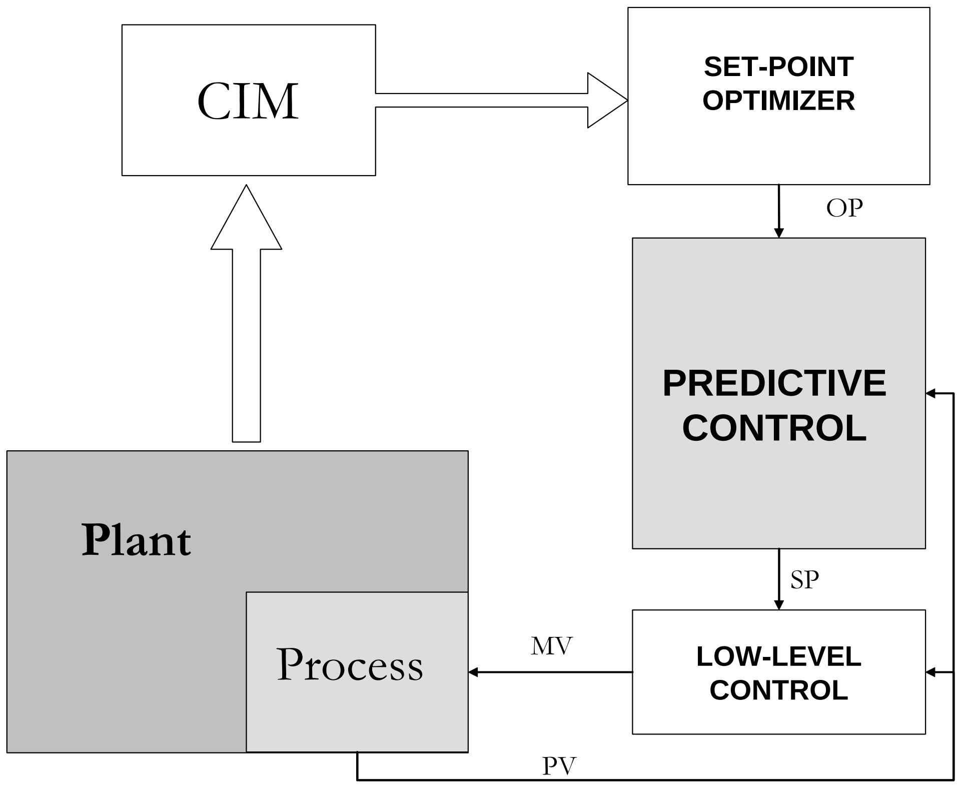


Figure 1.3: Integral Control Structure.

predictive control has achieved great results in the last years (Magni et al., 2001c; Limon et al., 2006a, 2009a), but its high computational burden makes this problem still open.

1.3 Model Predictive Control

One of the most successful control techniques for constrained systems is model predictive control (MPC). The main idea of MPC is to use a dynamic model of a system to forecast system behavior and, based on this prediction, take the best decision (Rawlings and Mayne, 2009). MPC is capable of ensuring an admissible evolution of the system while optimizing the closed-loop performance measured by a cost function that takes into account the error with the desired setpoint. The cost function is based on the prediction of the future evolution of the system by means of the prediction model of the form

$$x(j+1) = f(x(j), u(j))$$

The cost function usually considered is in the form:

$$V_N(x; \mathbf{u}) = \sum_{j=0}^{N-1} \ell(x(j), u(j)) + V_f(x(N)),$$

where $u(j)$ is the future sequence of control action computed at the current sampling time k, and $x(j)$ is the predicted state at sampling time k, considering that $x(0) = x$. The function $\ell(x, u)$ is known as stage cost, while the cost-to-go $V_f(x)$ is the terminal cost function.

The *best decision* is taken by optimizing the cost function: the optimal future sequence $\mathbf{u}^0$ of control actions is computed to minimize the predicted cost while satisfying the constraints. This optimization problem is mathematical programming problem that can be posed as follow:

$$\begin{aligned}
&\min_{\mathbf{u}} V_N(x, \mathbf{u}) \\
&s.t. \\
&\quad x(0) = x, \\
&\quad x(j+1) = f(x(j), u(j)), \\
&\quad u(j) \in U \qquad\qquad j = 0, \cdots, N-1 \\
&\quad x(j) \in X \qquad\qquad j = 0, \cdots, N-1 \\
&\quad x(N) \in \Omega.
\end{aligned}$$

The additional constraint on the terminal state is added for ensuring stability.

The feedback is achieved by means of the receding horizon technique: the first element of the optimal control sequence $\mathbf{u}^0$ is applied to the system and the optimization problem is re-computed at each sample time. Thus, the control law is given by

$$h(x) = u^0(0; x)$$

The optimal performance in MPC would be achieved if the whole predicted evolution of the system would be considered, but this leads to an infinite-horizon approach that cannot, in general, be solved. Thus, MPC considers a finite prediction horizon, which makes the problem tractable at the expense of the loss of the good properties of the optimal control problem, such as stability and inherent robustness. To overcome this problem, some additional conditions must be considered in the controller design (Mayne et al., 2000; Rawlings and Mayne, 2009).

There exist different stabilizing formulations of MPC:

- MPC with terminal equality constraint (Kwon and Pearson, 1977): the stability condition is adding an additional constraint over the state at the end of horizon $x(N)$ (terminal state) called terminal constraint:

  $$x(N) = x_s^*$$

  where x_s^* is the desired steady state.
- MPC with terminal cost (Bitmead et al., 1990): the stability condition is adding a new term to the cost function that penalizes the state at the end of the horizon.
- MPC with inequality terminal constraint (Michalska and Mayne, 1993): The terminal equality constraint is replaced by a set Ω that has to fulfil certain conditions.

  $$x(N) \in \Omega$$

  This approach provides a larger domain of attraction and less numerical problems than the equality constraint approach.
- MPC with terminal cost and constraint (Sznaier and Damborg, 1987): This approach is the result of the union of the last 2 techniques, adding a terminal cost to the cost function and using an inequality terminal constraint.

In (Mayne et al., 2000) and (Rawlings and Mayne, 2009) all these formulations are analyzed and it is established that adding a terminal cost together with a suitable terminal constraint has resulted to be essential to the stabilizing design. These conditions can be write down as follows:

Let Ω be a set in $\mathbb{R}^n$, let $V_f(x)$ be a positive definite function, continuous at the origin and let $h(x)$ be a control law such that,

- for all $x \in \Omega \subseteq X$, then $f(x, h(x)) \in \Omega$ and $h(x) \in U$
- for all $x \in \Omega$, we have that

  $$V_f(f(x, h(x)) - V_f(x) \leq -\ell(x, h(x))$$

Based on this, the optimal cost can be considered as a Lypaunov function: the invariant condition on terminal set Ω ensures feasibility of the closed-loop evolution of the system, while the condition on terminal function $V_f(x)$ guarantees convergence.

1.3.1 MPC and the setpoint changes

Usually, a higher level real time optimizer provides to the process plant a target or desired setpoint. If this operating point changes then the lower level control law must deal with this setpoint change. The classic solution is to translate the system to the new steady state (Muske and Rawlings, 1993). This solution guarantees the setpoint tracking when there are no constraints but may not solve the problem when the plant has constraints.

If the optimal control law is calculated by means of an infinite horizon regulator, any admissible setpoint can be tracked in an admissible way, for the nominal case. However, the computational effort to calculate an infinite horizon optimal control law is unaffordable due to the presence of constraints, and hence finite prediction horizons N are usually considered. In this case, the change of setpoints can produce a loss of feasibility of the optimization problem, mainly because the terminal region at the new setpoint could be unreachable in N steps, which makes the optimization problem unfeasible for not fulfilling the terminal constraint. Moreover, the terminal set calculated for a certain equilibrium point may not be an invariant set for the new setpoint. In this case, a re-calculation of an appropriate value of the prediction horizon is necessary to ensure feasibility. Therefore, this would require an on-line re-design of the controller for each setpoint, which can be computationally unaffordable.

Example 1.1 *Consider a LTI system given by:*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.0 & 0.5 \\ 1.0 & 0.5 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The system is constrained to $\|x\|_\infty \leq 5$ *and* $\|u\|_\infty \leq 0.3$. *Consider also an MPC with weighting matrices* $Q = I_2$ *and* $R = I_2$.

In figure 1.4 the loss of feasibility problem under a setpoint change is illustrated. Consider that the current state is x_0,*the current target is* r_1, *the set* $O_\infty(r_1)$ *is the maximal invariant set for the system controlled by* $u = K(x - x_1) + u_1$, *(where* (x_1, u_1) *is the steady state and control action for the system at setpoint* r_1*) which is the terminal* *constraint for an MPC controller with horizon $N = 3$; in this case, the domain of attraction of the controller is $X_3(r_1)$, drawn in dashed-dotted line. Suppose that the setpoint changes to r_2 at a certain sampling time. The first consequence is that set $O_\infty(r_1)$ translated to the steady state corresponding to r_2 is not an admissible invariant set, since the constraint would be clearly violated. This leads to a loss of feasibility. Consider, then a new controller with invariant set $O_\infty(r_1)$ and domain of attraction $X_3(r_2)$, plotted in dashed line. Due to the fact that x_0 is not into $X_3(r_2)$, the MPC with this horizon is unfeasible. In order to recover feasibility, the prediction horizon should be enlarged to $N = 6$.*

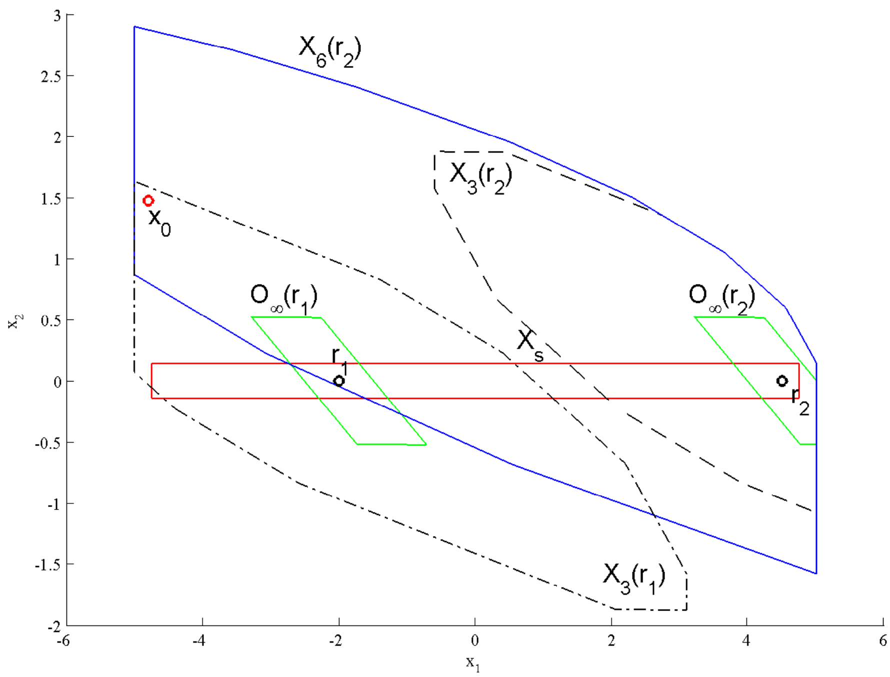


Figure 1.4: Loss of feasibility of the optimization problem derived from a non-admissible terminal condition or a short horizon.

This simple example shows how a setpoint change can produce a loss of feasibility derived from a non-admissible terminal condition or a short horizon.

1.3.2 MPC and the tracking problem

In order to overcome the loss of feasibility problem several solutions have been proposed: in (Rossiter et al., 1996; Chisci and Zappa, 2003) an auxiliary controller that is able to recover feasibility in finite time is used leading to a switching strategy. The controllers proposed in (Pannocchia and Rawlings, 2003; Pannocchia, 2004; Pannocchia and Kerrigan, 2005) consider the change of the setpoint as a disturbance to be rejected; thus, this technique is able to steer the system to the desired setpoint, but only when the variations of the setpoint are small enough, providing a conservative solution.

A different approach has been proposed in the context of the reference governors (Gilbert et al., 1999; Bemporad et al., 1997). This control technique assumes that the system is robustly stabilized by a local controller, and a nonlinear filtering of the reference is designed to ensure the robust satisfaction of the constraints. These controllers ensure robust tracking without considering the performance of the obtained controller nor the domain of attraction.

The problem of tracking in the case of nonlinear MPC has been considered in many works in literature. In (Findeisen et al., 2000) a nonlinear predictive control for setpoint families is presented, which considers a pseudolinearization of the system and a parametrization of the setpoints. The stability is ensured thanks to a quasi-infinite nonlinear MPC strategy, but the solution of the tracking problem is not considered. In (Magni et al., 2001b) an output feedback receding horizon control algorithm for nonlinear discrete-time systems is presented, which solves the problem of tracking exogenous signals and asymptotically rejecting disturbances generated by a properly defined exosystem. In (Magni and Scattolini, 2005) an MPC algorithm for nonlinear systems is proposed, which guarantees local stability and asymptotic tracking of constant references. This algorithm need the presence of an integrator preliminarily plugged in front of the system to guarantee the solution of the asymptotic tracking problem. In (Magni and Scattolini, 2007) an MPC algorithm for continuous-time, possibly non-square nonlinear systems is presented. The algorithm guarantees the tracking of asymptotically constant reference signals by means of a control scheme were the integral action is directly imposed on the error variables rather than on the control moves.

In (Limon et al., 2008a) a novel MPC for tracking is proposed, which is able to lead the system to any admissible setpoint in an admissible way. The main characteristics of this controller are: an artificial steady state considered as a decision variable, a cost that penalizes the error with the artificial steady state, an additional term that penalizes the deviation between the artificial steady state and the target steady state (the so-called *offset cost function*) and an extended terminal constraint, the invariant set for tracking. This controller ensures that under any change of the steady state target, the closed-loop system maintains the feasibility of the controller, converging to the target if admissible. The additional ingredients of the controller have demonstrated to affect the closed-loop performance of the controlled system (Alvarado, 2007).

1.4 Robustness in Model Predictive Control

It is well known that under mild conditions, MPC is able to ensure some degree of robustness (Scokaert et al., 1997; De Nicolao et al., 1996; Limon et al., 2002). When the uncertainties are big enough, a robust design must be accomplished. To this aim, an uncertainty model must be considered; this is typically considered as an external disturbance acting on the dynamics in a parametric way or by means of an additive term in the model function.

The traditional way of dealing with uncertainties in MPC is considering all their possible realizations in the formulation of the optimization problem. In this way, the constraints on state and input have to be fulfilled in a robust way, that is taking into account all their possible realizations throughout the predicted evolution of the system. The cost function, hence, can be based on nominal predictions, or can take into account the uncertainties considering the worst-case scenario. This is the main idea of the so-called min-max formulation (Fontes and Magni, 2003; Limon et al., 2006a; Mayne, 2001; Raimondo et al., 2009). Another way to deal with uncertainties is by adding, in the stage cost, an extra term that penalizes the possible uncertainties, like in the H_∞ formulation (Magni et al., 2001c).

These solutions for the robust problem are conservative solutions, due to the open-loop prediction that characterize MPC. To this aim, in literature there are lots of work based proposing a closed-loop formulation (Scokaert and Mayne, 1998; Lee and Yu, 1997; Kerrigan and Maciejowski, 2004; Mayne et al., 2006). In this formulation, the control problem is based on the calculation of a sequence of control laws, which however gives a infinite-dimensional optimization problem. As a consequence, this solution seems to be only theoretical (Mayne et al., 2000).
A particular formulation of closed-loop robust MPC, is the one proposed in (Kothare et al., 1996). In this formulation, the disturbed plant is considered as a convex combi-

nation of linear plants, and hence the control problem consist in finding a linear control law that stabilizes all the plants.

A trade-off solution between the open and the closed-loop formulations is to add a robustly pre-stabilized plant (Bemporad, 1998a; Chisci et al., 2001). This solution enhances robustness, while the optimization problem can be cast as a mathematical programming problem similar to the open-loop formulation. A recent novel robust MPC of this class based on the notion of a tube of trajectories has been proposed (Langson et al., 2004). Using this notion, an enhanced tube-based robust MPC controller has been proposed in (Mayne et al., 2005). This controller exploits the notion of invariant sets to obtain a robust control law based on nominal predictions.

Recently, in (Limon et al., 2009a), input-to-state stability (ISS) has been presented as a unified framework for the analysis of the stabilizing properties of MPC in presence of disturbances.

1.5 The zone control problem

In many cases, the optimal economic steady state operating condition is not given by a point in the output space (fixed set-point), but is a region into which the output should lie most of the time. In general, based on operational requirements, process outputs can be classified into two broad categories: (i) set-point controlled, outputs to be controlled at a desired value, and (ii) set-interval controlled, outputs to be controlled within a desired range. For instance, production rate and product quality may fall into the first category, whereas process variables, such as level, pressure, and temperature in different units/streams may fall into the second category. There are many reasons for using set-interval control in real applications. One reason can be the necessity of let the output lying in a zone, for some economic reason. Another one may be the presence of too much controlled outputs, and a few number of manipulated variables to control them. Conceptually, the output intervals are not output constraints, since they are steady state desired zones that can be transitorily disregarded, while the (dynamic) constraints must be respected at each time. In addition, the determination of the output intervals is related to the steady state operability of the process, and it is not a trivial problem. An important aspect is the compatibility between the available input set (given by the input constraints) and the desired output set (given by the output intervals). In (Vinson and Georgakis, 2000) and (Lima and Georgakis, 2008), for instance, an operability index that quantify how much of the region of the desired outputs can be achieved using the available inputs, taking into account the expected disturbance set, is defined. As a result a methodology to obtain the tightest possible operable set of achievable output steady state is derived. Then, the operating control intervals should be subsets of these tightest intervals. In practice, however, the operators are not usually aware of these maximum zones and may select control zones that are not fully consistent with the maximum zones and the operating control zones may be fully or partly unreachable. The MPC controller has to be robust to this poor selection of the control zones.

From the point of view of the controller, several approaches have been developed to account for the set-interval control. (Qin and Badgwell, 2003), describes a variety of industrial controller and mentions that they always provide a zone control option. That paper presents two ways to implement zone control: 1) defining upper and lower soft constraints, and 2) using the set-point approximation of soft constraints to implement the upper and lower zone boundaries (the DMC-plus algorithm). One of the main problems of these industrial controllers (as was stated in the same paper) is the lack of nominal stability. A second example of zone control can be found in (Zanin et al., 2002), where the authors exemplify the application of this strategy to a FCC system. Although this strategy has shown to have an acceptable performance, stability cannot be proved, even if an infinite horizon is used, since the control system keeps switching from one controller to another throughout the continuous operation of the process. A third example is the closed-loop stable MPC controller presented in (Gonzalez and Odloak, 2009). In this approach, the authors develop a controller that considers the zone control of the system outputs and incorporates steady state economic targets in the control cost function. Assuming open-loop stable systems, classical stability proofs are extended to the zone control strategy by considering the output set-points as additional decision variables of the control problem. Furthermore, a set of slack variables is included into the formulation to assure both, recursive feasibility of the on-line optimization problem and convergence of the system inputs to the targets. This controller, however, is formulated for stable open-loop stable systems, and since it considers a null controller as local controller, it does not achieve local optimality. An extension of this strategy to the robust case, considering multi-model uncertainty, was proposed in (González et al., 2009).

1.6 Optimization of process economic performance

As already discussed in section 1.2, the standard structure of all industrial advanced control systems is characterized by a two-layer structure. The first level performs a steady state optimization, and it is usually called as Real Time Optimizer (RTO). The RTO determines the optimal setpoints and sends them to the second level, the advanced control systems, which performs a dynamic optimization. In many control process, MPC is the advances control formulation chosen for this level (Rawligns and Amrit, 2009).

The issue of this structure are related on the role of the RTO. The real-time optimizations that this level performs are usually based on the stationary model of the plant. Each sample time the economic criterion is optimized, in order to achieve the best value of the steady state variables for the stationary plant. The result if this optimization is passed to the advanced controller as a setpoint. The problem is that, usually, this setpoint results to be inconsistent or unreachable with respect to the dynamic layer, and this happens mainly because of the discrepancies between the stationary model of the RTO and the dynamic model used for regulation. In (Rao and Rawlings, 1999) the authors propose some methods for solving this problems and finding a reachable steady state as close as possible to the unreachable setpoint provided by the RTO.

An hot topic in in control literature is nowadays the design economic controllers. This kind of controllers are defined economic because the optimization performed by the RTO is not a standard steady state optimization based on the dynamic of the system. What the RTO minimizes is an economical criterion based on some aspects like the demand of production of the process. Hence, the optimal setpoint calculated by the RTO may not coincide with the dynamic steady state of the system (Kadam and Marquardt, 2007).

The interest of (Rawligns and Amrit, 2009) is that it pointed out the advantage of using the advanced control layer - the dynamic MPC lawyer- to perform the economic optimization. They first explore the case of unreachable setpoints and show that sometimes, it is better no to reach the steady state quickly. They also consider the case of replacing the setpoint objective function with a cost function minimizing some economic criterion. A stable MPC for the case of unreachable setpoints is presented in (Rawlings et al., 2008), while in (Würth et al., 2009) a stable infinite-horizon (nonlinear) economic controller is presented. Stability using the standard framework of Lyapunov function is proved in (Diehl et al., 2011) and (Huang et al., 2011).

1.7 Large scale systems

In the process industries, plants are usually considered as large scale systems, consisting of linked unit of operations. Therefore, they can be divided into a number of subsystems, connected by networks of different nature, such as material, energy or information streams (Stewart et al., 2010). The overall control of these plants by means of a centralized controller - a single agent controlling all subsystems - is difficult to realize. The issue is not just a computational problem. Nowadays, the increased computational power, faster optimization solver, and specific algorithms designed for large scale systems, makes the centralized control task realizable (Bartlett et al., 2002; Pannocchia et al., 2007). Since each subsystem undertakes a different task, in order to achieve its optimal economic performance, sometimes it has to disregard other subsystems information. In some other case, the interchange of information between subsystem results to be important in order to achieve optimal performance. The real issue is the organization and the maintenance of centralized controllers. Another common way to control an overall plant is given by decentralized controller. In this formulation, each subsystem is controlled independently, without interchange of information between different subsystems. The information that flows in the network is usually considered as a disturbance by each subsystem (Huang et al., 2003; Sandell Jr. et al., 1978; Raimondo et al., 2007a; Magni and Scattolini, 2006). The drawback of this control formulation is the big loss of information when the interaction between subsystems are strong (Cui and Jacobsen, 2002).

An hot topic in the control community is nowadays distributed control, that is, a control strategy based on different agents - instead of a centralized controller - controlling each subsystems, which may or may not share information. There are different distributed control strategies proposed in literature. The difference between these distributed control strategies is in the use of this open-loop information, allowing to define basically to kind of distributed control formulations: noncooperative controllers and cooperative controllers. In noncooperative controllers, each agent makes decision on the single subsystem considering the other subsystems information only locally (Camponogara et al., 2002b; Dunbar, 2007). This strategy is usually referred as noncooperative dynamic game, and the performance of the plant converge to a Nash equilibrium (Başar and Olsder, 1999). Cooperative distributed controllers on the other hand, consider the effect of all the control actions on all subsystems in the network (Venkat, 2006; Pannocchia et al., 2009; Stewart et al., 2010). Each controller optimize an overall plant object function, such as the centralized object. Cooperative control makes the system converging to the Pareto optimum, that is the centralized performance. Cooperative control is a form of suboptimal control for the overall plant, and therefore stability is proved resorting to suboptimal control theory (Stewart et al., 2010; Scokaert et al., 1999).

MPC is one of the most used control structure to cope with distributed control (Rawlings and Mayne, 2009, Chapter 6). In (Magni and Scattolini, 2006) an MPC approach for nonlinear systems is proposed, where no information is exchanged between the local controllers. An input-to-state stability proof for this approach is given in (Raimondo et al., 2007b). In (Liu et al., 2009, 2008) the authors present a controller for networked nonlinear systems, which is based on a Lyapunov-based model predictive control. In (Venkat et al., 2007; Stewart et al., 2010) a cooperative distributed MPC is presented, in which suboptimal input trajectories are used to stabilize the plant.

1.8 Contributions of this thesis

The objective of this thesis is to study the tracking problem in model predictive control for linear and nonlinear systems, analyzing in particular some issues like loss of feasibility, optimality, economic optimality, zone control problems. In particular the MPC for tracking (Limon et al., 2008a) is studied and extended to some control problems like optimal MPC, zone control, nonlinear MPC, economic MPC, distributed MPC.

1.8.1 MPC for tracking with optimal closed-loop performance

In chapter 2, an enhanced formulation of the MPC for tracking (Limon et al., 2008a) is presented. The proposed controller inherits the main ingredients from the MPC for tracking (Limon et al., 2008a), which are:

- An artificial steady state considered as a decision variable.
- A cost function that penalizes the error with the artificial steady state.

- An additional term that penalizes the deviation between the artificial steady state and the target, (the so-called *offset cost function*).

- An invariant set for tracking considered as extended terminal constraint.

In this chapter, the MPC for tracking is extended considering a general offset cost function. Under some sufficient conditions, this function ensures the local optimality property, letting the controller achieve optimal closed-loop performance. Moreover, the chapter presents a characterization of the region of local optimality and a non-expensive way to calculate it.

Besides, this novel formulation allows to consider any set of process variables as target which makes the controller suitable for non-square plants. Furthermore, the proposed MPC for tracking deals with the case that the target to track does not fulfil the hard constraints or this is not an equilibrium point of the linear model. In this case the proposed controller steers the system to an admissible steady state (different to the target) which minimizes the offset cost function. This property means that the offset cost function plays the same role as a real time optimizer, which is built in the proposed MPC.

1.8.2 MPC for tracking target sets

In chapter 3 the application of MPC for tracking to the zone control problems, is presented. The problem is addresses by designing an MPC for tracking a certain set, not a fixed point. To this aim, the concept of distance to a set is introduced and exploited for the design of the MPC control law. The proposed controller ensures recursive feasibility and convergence to the target set for any stabilizable plant. This property holds for any class of convex target sets and also in the case of time-varying target sets. For the case of polyhedral target sets, several formulations of the controller are proposed that allows to derive the control law from the solution of a single quadratic programming problem. One of these formulations allows also to consider target points and target sets simultaneously in such a way that the controller steers the plant to the target point if reachable while it steers the plant to the target set in the other case.

1.8.3 Robust MPC for tracking based on nominal predictions

Chapter 4 deals with the problem of robust tracking of uncertain linear systems. A robust MPC based on nominal predictions is presented. The controller presented in (Ferramosca et al., 2010a) and chapter 3 is extended to cope with the problem of robust tracking of target sets in presence of additive disturbance. The proposed controller uses the results presented in (Chisci et al., 2001), in which an MPC based on nominal predictions and restricted constraints is presented, which ensures stability, robust satisfaction of the constraints and recursive feasibility. The plant is assumed to be modeled as a linear system with additive uncertainties confined to a bounded known polyhedral set. Under mild assumptions, the proposed MPC is feasible under any change of the controlled variables target and steers the uncertain system to (a neighborhood of) the target if this is admissible. If the target is not admissible, the system is steered to the closest admissible operating point.

1.8.4 Distributed MPC for tracking

In chapter 5 a distributed MPC for tracking control strategy for constrained linear system is presented. In particular the MPC for tracking presented in chapter 2 is extended to the case of large scale distributed systems.

Among the different solutions presented in literature, this chapter particularly focuses on the cooperative formulation for distributed MPC presented in (Rawlings and Mayne, 2009, Chapter 6), in (Venkat, 2006) and in (Stewart et al., 2010). In this formulation, the players share a common objective, which can be considered as the overall plant objective. This means that any player calculates its corresponding inputs by minimizing the same and unique cost function, by means of an iterative (and hence suboptimal) distributed optimization problem. Stability is proved by means of suboptimal MPC theory (Scokaert et al., 1999). Convergence to the centralized optimal target and recursive feasibility after any change of the operation point, are guaranteed by means of a centralized target problem solution and the use of a specific *warm start* algorithm.

1.8.5 MPC for tracking constrained nonlinear systems

Chapter 6 deals with the problem of design a MPC control strategy for tracking in case of constrained nonlinear systems.

The controller presented in this chapter inherits the main features of the one presented in chapter 2. Particular interest presents, in this context, the calculation of the terminal ingredients. Three formulations of the controller are presented, which consider respectively the cases of terminal equality constraint, terminal inequality constraint and no terminal constraint.

As for the case of terminal inequality constraint, in particular, a method for the calculation of the terminal constraint is proposed, based on the LTV modeling technique and the partition method proposed in (Wan and Kothare, 2003a,b). The idea is to design a set of local predictive controllers, whose feasible regions cover the entire steady state manifold.

1.8.6 Economic MPC for a changing economic criterion

Recently, a new MPC formulation aimed to consider an economic performance stage cost instead of a tracking error stage cost, has been proposed in (Rawlings et al., 2008; Diehl et al., 2011). In (Rawlings et al., 2008; Diehl et al., 2011) the authors show that this controller is stable and asymptotically steers the system to the economically optimal admissible steady state, and that the controlled system exhibits better performance with respect to the setpoint than standard target-tracking MPC formulations.

If the economic criterion changes, the economically optimal admissible steady state where the controller steers the system may change, and the feasibility of the controller may be lost. In chapter 7, an economic MPC for a changing economic criterion is presented. This controller result to be an hybrid formulation of both the MPC for tracking (Limon et al., 2008a; Ferramosca et al., 2009a) and the economic MPC (Rawlings et al., 2008; Diehl et al., 2011), since it inherits the feasibility guarantee of the MPC for tracking and the optimality with respect to the setpoint of the economic MPC.

1.9 List of Publications

1.9.1 Book Chapters:

1. D. Limon, A. Ferramosca, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking of constrained nonlinear systems. *Book of the* 3^{rd} *International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, pp. 315-323. 2009.

2. D. Limon, T. Álamo, D.M. Raimondo, D. Muñoz de la Peña, J.M. Bravo, A. Ferramosca, and E. F. Camacho. Input-to-State stability: a unifying framework for robust Model Predictive Control. *Book of the* 3^{rd} *International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, pp. 1-26. 2009.

1.9.2 Journal Papers:

1. A. Ferramosca, D. Limon, A.H. González, D. Odloak, and E. F. Camacho. MPC for tracking zone regions. *Journal of Process Control*, 20 (4), pp. 506-516. 2010.

2. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking with optimal closed-loop performance. *Automatica*, 45 (8), pp. 1975-1978. 2009.

3. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, F. Castaño, and E. F. Camacho. Optimal MPC for tracking of constrained linear systems. *International Journal of Systems Science*, 42 (8). To be published in August 2011.

1.9.3 International Conference Papers:

1. A. Ferramosca, D.Limon, J.B. Rawlings, and E.F. Camacho. Cooperative distributed MPC for tracking. *Proceedings of the* 18^{th} *IFAC World Congress*, 2011.

2. A. Ferramosca, J.B. Rawlings, D.Limon, and E.F. Camacho. Economic MPC for a changing economic criterion. *Proceedings of* 49^{th} *IEEE Conference on Decision and Control, (CDC)*, 2010.

3. D. Limon, I. Alvarado, A. Ferramosca, T. Álamo, and E. F. Camacho. Enhanced robust NMPC based on nominal predictions. *Proceedings of* 8^{th} *IFAC Symposium on Nonlinear Control Systems, (NOLCOS)*. 2010.

4. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking of constrained nonlinear systems. *Proceedings of* 48^{th} *IEEE Conference on Decision and Control, (CDC)*, 2009.

5. A. Ferramosca, D. Limon, A.H. González, D. Odloak, and E. F. Camacho. MPC for tracking target sets. *Proceedings of* 48^{th} *IEEE Conference on Decision and Control, (CDC)*, 2009.

6. A. Ferramosca, D. Limon, F. Fele, and E. F. Camacho. L-Band SBQP-based MPC for supermarket refrigeration systems. *Proceedings of* 10^{th} *European Control Conference, (ECC)*, 2009.

7. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking with optimal closed-loop performance. *Proceedings of* 47^{th} *IEEE Conference on Decision and Control, (CDC)*, 2008.

8. D. Limon, A. Ferramosca, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking of constrained nonlinear systems. *Proceedings of the* 3^{rd} *International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, 2008.

9. I. Alvarado, D. Limon, A. Ferramosca, T. Álamo, and E. F. Camacho. Robust tube-based MPC for tracking applied to the quadruple tank process. *Proceedings of the IEEE International Conference on Control Applications, (CCA)*. 2008.

10. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. Optimal MPC for tracking of constrained linear systems. *Proceedings of* 8^{th} *Portuguese Conference on Automatic Control, (CONTROLO)*. 2008.

1.9.4 National Conference Papers:

1. A. Ferramosca, I. Alvarado, D. Limon, and E. F. Camacho. MPC para el seguimiento del ángulo de cabeceo de un helicóptero. *Proceedings of* 28^{th} *Jornadas de Automática*. 2007.

Chapter 2

MPC for tracking with optimal closed-loop performance

2.1 Introduction

Most of the MPC stability and feasibility results consider the regulation problem, that is steering the system to a fixed steady state (typically the origin). It is clear that for a given non-zero setpoint, a suitable choice of the steady state can be chosen and the problem can be posed as a regulation problem translating the state and input of the system (Muske and Rawlings, 1993). However, since the stabilizing choice of the terminal cost and constraints depends on the desired steady state, when the target operating point changes, the feasibility of the controller may be lost and the controller fails to track the reference (Rossiter et al., 1996; Bemporad et al., 1997; Pannocchia and Kerrigan, 2005; Alvarado, 2007), thus requiring to re-design the MPC at each change of the reference. The computational burden that the design of a stabilizing MPC requires may make this approach not viable. For such case, the steady state target can be determined by solving an optimization problem that determines the steady state and input targets. This target calculation can be formulated as different mathematical programs for the cases of perfect target tracking or non-square systems (Muske, 1997), or by solving a unique problem for both situations (Rao and Rawlings, 1999). In (Limon et al., 2008a) an MPC for tracking is proposed, which is able to lead the system to any admissible setpoint in an admissible way. The main characteristics of this controller are: an artificial steady state is considered as a decision variable, a cost that penalizes the error with the artificial steady state is minimized, an additional term that penalizes the deviation between the artificial steady state and the target steady state is added to the cost function (the so-called *offset cost function*) and an invariant set for tracking is considered as extended terminal constraint. This controller ensures that under any change of the steady state target, the closed-loop system maintains the feasibility of the controller and ensures the convergence to the target if admissible.

However, some problems still remain open in the formulation of the MPC for tracking. These are mainly two: the potential loss of the optimality property due to the addition of the artificial steady state together with the proposed cost function and the convergence of the closed-loop system when the target is not reachable (due to the constraints and/or inconsistency with the equilibrium point equation). In this chapter, the MPC for tracking is extended considering a general offset cost function. Under some mild sufficient conditions, this function ensures the local optimality property, letting the controller achieve optimal closed-loop performance.
Besides, this novel formulation allows to consider any set of process variables as target which makes the controller suitable for non-square plants. Furthermore, the proposed MPC for tracking deals with the case that the target to track does not fulfil the hard constraints or this is not an equilibrium point of the linear model. In this case this control law steers the system to an admissible steady state (different to the target) which minimizes the offset cost function. This property means that the offset cost function plays the same role that the cost function of a steady state target optimizer which is built in the proposed MPC.

2.2 Problem Description

Let a discrete-time linear system be described by:

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the current state of the system, $u \in \mathbb{R}^m$ is the current input, $y \in \mathbb{R}^p$ is the controlled output and x^+ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u})$, $j \in \mathbb{I}_{\geq 0}$, where $x = \phi(0; x, \mathbf{u})$. Note that no assumption is considered on the dimension of the states, inputs and outputs and hence non square systems (namely $p > m$ or $p < m$) might be considered.

The controlled output is the variable used to define the target to be tracked by the controller. Since no assumption is made on matrices C and D, these variables might be (a linear combination of) the states, (a linear combination of) the inputs or (a linear combination of) both.

The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively. The system is subject to hard constraints on state and control:

$$(x(k), u(k)) \in \mathcal{Z}$$

for all $k \geq 0$. $\mathcal{Z} \subset \mathbb{R}^{n+m}$ is a compact convex polyhedron containing the origin in its interior.

Assumption 2.1 *The pair (A,B) is stabilizable and the state is measured at each sampling time.*

The problem we consider is the design of an MPC controller $\kappa_N(x)$ to track a (possible time-varying) target output y_t. If y_t is an admissible steady output (that is, the corresponding operation point fulfils the constraints), the closed loop system evolves to this target without offset. If y_t is not consistent with the linear model considered for predictions, namely, it is not a possible steady output of system (2.1) or this is not admissible, the closed-loop system evolves to an admissible steady state which minimizes a given performance index.

2.3 Preliminary results

The MPC for tracking (Limon et al., 2008a; Alvarado, 2007) is capable to ensure feasibility under any change of setpoint due to the use of three main ingredients: the artificial reference, the so-called *offset cost* function and the invariant set for tracking.

In this section, the meaning and the role of these ingredients will be introduced.

2.3.1 Characterization of the steady state of the system

In this work, as in (Limon et al., 2008a), the term *artificial reference* will denote an admissible equilibrium point of the system, which is used as auxiliary reference to track, due to the constrained optimal control problem solved at any time instant k.

Consider a given steady output y_t. The aim of this section is to derive a characterization of the steady state and input (x_s, u_s) which provides the desired output (if admissible), i.e. $y_t = Cx_s + Du_s$. If y_t is not a possible steady output of system (2.1) or it is not admissible, the steady state of the system is determined by a steady output that is the minimizer of a certain performance index, the *offset cost* function. This function will be introduce in the next section.

Under assumption 2.1, any steady state and input of system (2.1) associated to y_t must satisfy the following equation:

$$\begin{bmatrix} A - I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n,1} \\ y_t \end{bmatrix} \tag{2.2}$$

Denoting $z_s = (x_s, u_s)$ and

$$E = \begin{bmatrix} A - I_n & B \\ C & D \end{bmatrix}, \quad F = \begin{bmatrix} \mathbf{0}_{n,p} \\ I_p \end{bmatrix} \tag{2.3}$$

equation (2.2) can be written as

$$Ez_s = Fy_t$$

In (Limon et al., 2008a) the authors state that the steady state and input (x_s, u_s) of the system can be parameterized as a linear combination of a vector $\theta \in \mathbb{R}^m$, that is

$$(x_s, u_s) = M_\theta \theta \tag{2.4}$$

where matrix M is such that

$$[A - I_n \, B] M_\theta = 0$$

The steady controlled outputs are given by

$$y_t = N_\theta \theta \tag{2.5}$$

where $N_\theta = [C \, D] M_\theta$.

The dimension of θ is m, which is the dimension of the subspace of steady states and inputs that can be parameterized by a minimum number of variables. Hence, equation (2.4) represents a mapping of (x_s, u_s) and y_t onto the subspace of θ. The set of setpoints y_t that can be admissibly reached is the subspace spanned by the columns of N_θ. Then this set depends on the rank of matrix E. Defining as r the rank of matrix E and as r_p the rank of matrix N_θ, we have these two cases:

1. If $r = n + p$, then $r_p = p$. Hence, the system can be steered to any setpoint y_t.

2. If $r < n + p$ then $r_p < p$. This implies that equation (2.2) has a solution only for those setpoints y_t contained in the linear subspace spanned by the columns of N_θ, and hence not every reference y_t can be reached. The usual way of overcoming this problem is re-defining the system: *new* controlled variables, $y_c \in \mathbb{R}^{p_c}$ with $p_c \leq r_p$ are taken; these *new* controlled variables are chosen as a linear combination of the actual outputs, i.e. $y_c = L_c y = L_c Cx + L_c Du$. Matrix L_c must be such that the rank of the *new* matrix

$$E_c = \begin{bmatrix} A - I_n & B \\ L_c C & L_c D \end{bmatrix}$$

is full row rank, i.e. its rank is $n + p_c$.

Since the system is constrained, it should be steered to those steady states that satisfy the constraints. The set of these admissible steady states and inputs is defined as

$$\mathcal{Z}_s = \{z = (x, u) : \; z \in \mathcal{Z}, \text{and } (A - I_n)x + Bu = 0\}$$

Thus, the set of admissible steady states and the set of admissible inputs is defined as

$$X_s = Proj_x(\mathcal{Z}_s),\; U_s = Proj_u(\mathcal{Z}_s)$$

respectively. The set of all admissible setpoints is denoted as $\mathcal{Y}_s$ and it is given by

$$\mathcal{Y}_s = \{y \in \mathbb{R}^p : \; \exists z_s = (x_s, u_s) \in \lambda\mathcal{Z}_s \text{ such that } y = Cx_s + Du_s\}$$

with $\lambda \in (0, 1)$. For a given admissible setpoint $y_t \in \mathcal{Y}_s$, the steady state and input, i.e. z_s, such that the corresponding output is equal to y_t is unique if and only if the rank of E is equal to $n + m$. If the rank of E is less than $n + m$, then there exists infinite steady states and inputs z_s such that the associated output is equal to y_t.

2.3.2 The offset cost function

In this section the *offset cost* function and its role in the MPC for tracking are presented.

In the MPC for tracking (Limon et al., 2008a; Alvarado, 2007), in order to ensure the feasibility of the problem for any desired setpoint, an artificial steady augmented state $z_s = (x_s, u_s) = M_\theta \theta$ is introduced as a decision variable in the minimization of the performance index. This means that, at any time k the controller finds an optimum steady state (x_s, u_s) to which the system can converge maintaining the feasibility of the problem. Convergence to the desired setpoint is ensured by adding a term $V_O = \|\theta - \theta_t\|_T^2$ in the cost function (*offset cost*) that penalizes the deviation between the desired steady state (θ_t) and the artificial one (θ).

In this work, a new formulation of the *offset cost* function is used. For reason that will be clear in the following of this dissertation, the formulation of $V_O(.)$ as a square of a norm has been changed in a general convex offset cost function. As it will be demonstrated later on, under mild assumptions, this function provides significant properties to the controlled system.

2.3.3 Calculation of the invariant set for tracking

In this section, the calculation of the terminal ingredients of the MPC for tracking (Limon et al., 2008a; Alvarado, 2007), the invariant set for tracking, is presented.

Consider the following controller

$$u = K(x - x_s) + u_s, \tag{2.6}$$

where (x_s, u_s) is a desired steady state. It is well known that if the controller gain K stabilize the closed-loop system, that is $A + BK$ has all its eigenvalues inside the unit circle, then the system is steered to the desired steady state. Since the system is constrained, this controller leads to an admissible evolution of the system only in a neighborhood of the steady state.

Substituting (2.4) in (2.6), it results that

$$\begin{aligned} u &= Kx + [-K \; I_m] \begin{bmatrix} x_s \\ u_s \end{bmatrix} \\ &= Kx + [-K \; I_m] M_\theta \theta \\ &= Kx + L\theta \end{aligned}$$

where $L = [-K \; I_m] M_\theta \in \mathbb{R}^{m \times m}$. Consider the augmented state $x_a = (x, \theta)$, then the closed-loop augmented system can be defined by the following equation

$$\begin{bmatrix} x \\ \theta \end{bmatrix}^{+} = \begin{bmatrix} A+BK & BL \\ 0 & I_{n_\theta} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{2.7}$$

that is, $x_a^+ = A_a x_a$.

Define the following convex polyhedron for a given $\lambda \in (0,1)$

$$\mathcal{X}_\lambda^a = \{x_a = (x,\theta) : z = (x, Kx + L\theta) \in \mathcal{Z},\ z_s = M_\theta \theta \in \lambda \mathcal{Z}\}$$

It is clear that the set of constraints for system (2.7) is $\mathcal{X}^a = \mathcal{X}^a_{\lambda=1}$. That is, both $(x,u) = (x, Kx+L\theta)$ and $(x_s,u_s) = M_\theta\theta$ must belong to $\mathcal{Z}$.

We say that a set Ω_t^a is an admissible invariant set for tracking, for system (2.7) constrained to $\mathcal{X}^a$, if $\forall x_a \in \Omega_t^a$, then $A_a x_a \in \Omega_t^a$ and $\Omega_t^a \subseteq \mathcal{X}^a$. By definition, the maximal admissible invariant set for tracking is given by:

$$\mathcal{O}_\infty^a = \{x_a : A_a^i x_a \in \mathcal{X}^a, \forall i \geq 0\}$$

Due to the unitary eigenvalues of A_a, this set might be not finitely determined, i.e., described by a finite set of constraints (Gilbert and Tan, 1991). Hence, consider the maximal admissible invariant set for tracking evaluated using $\mathcal{X}_\lambda^a$ as constraint set, which is given by

$$\mathcal{O}_{\infty,\lambda}^a = \{x_a : A_a^i x_a \in \mathcal{X}_\lambda^a, \forall i \geq 0\}$$

Taking into account that the controller given by (2.6) guarantees that (x,u) converges asymptotically to (x_s,u_s) and following similar arguments to (Gilbert and Tan, 1991), it can be shown that for all $\lambda \in (0,1)$, $\mathcal{O}_{\infty,\lambda}^a$ is finitely determined and $\lambda\mathcal{O}_\infty^a \subset \mathcal{O}_{\infty,\lambda}^a \subset \mathcal{O}_\infty^a$. Notice that because λ can be chosen arbitrarily close to 1, the obtained invariant set can be made arbitrarily close to the real maximal invariant set $\mathcal{O}_\infty^a$.

In what follows, superscript a denotes that set Ω_t^a is defined in the extended state, while no superscript denotes that set Ω_t is defined in the state vector space x, i.e., $\Omega_t = Proj_x(\Omega_t^a)$.

Hereafter $\mathcal{O}_\infty(x_s)$ denotes the maximal invariant set of states that can be steered to x_s in an admissible way by the control law (2.6). It is easy to see that the computed polyhedral set $\mathcal{O}_{\infty,\lambda}$ is such that

$$\mathcal{O}_{\infty,\lambda} = \bigcup_{x_s \in \lambda X_s} \mathcal{O}_\infty(x_s)$$

It is clear that set λX_s is contained in $\mathcal{O}_{\infty,\lambda}$.

2.4 Enhanced Formulation of the MPC for tracking

In this section we present a novel formulation of the MPC for tracking which generalizes and improves the one presented by the authors in (Limon et al., 2008a) and (Alvarado, 2007). This new formulation maintains the main ingredients of the previous one:

(i) an artificial steady state and input is considered as decision variables

(ii) the stage cost penalizes the deviation of the predicted trajectory with the artificial steady conditions

(iii) an *offset cost* function is added to penalize the deviation between the artificial steady state and the target setpoint

(iv) the invariant set for tracking is considered as extended terminal constraint.

In this work, this controller is extended to the case of considering a general offset cost function $V_O(\cdot)$ defined as follows:

Definition 2.2 *Let the offset cost function $V_O : \mathbb{R}^p \to \mathbb{R}$ be a convex, positive definite and subdifferentiable function such that $V_O(0) = 0$ and such that the minimizer of*

$$\min_{y_s \in \mathcal{Y}_s} V_O(y_s - y_t)$$

is unique.

As it will be shown in the next section, this formulation can provide optimal closed-loop performance to the controller.

Remark 2.3 *Notice that a subdifferentiable function (Boyd and Vandenberghe, 2006) is a function that admits subgradients. Given a function f, g is a subgradient of f at x if*

$$f(y) \geq f(x) + g'(y - x) \quad \forall y$$

Notice also that, the term subdifferential defines the set of all subgradients of f at x and is noted as $\partial f(x)$. This set is a nonempty closed convex set.

The proposed cost function of the MPC is given by:

$$V_N(x;\mathbf{u},\theta) \;=\; \sum_{j=0}^{N-1} \|x(j)-x_s\|_Q^2+\|u(j)-u_s\|_R^2 + \|x(N)-x_s\|_P^2+ V_O(y_s-y_t)$$

where $x(j)$ denotes the prediction of the state j-samples ahead, the pair $(x_s,u_s)=M_\theta\theta$ is the artificial steady state and input and $y_s=N_\theta\theta$ the artificial output, all of them parameterized by θ; y_t is the target of the controlled variables. The controller is derived from the solution of the optimization problem $P_N(x)$ given by

$$\begin{aligned}
V_N^0(x) \;&=\; \min_{\mathbf{u},\theta} V_N(x;\mathbf{u},\theta)\\
s.t. \;\; & x(0)=x,\\
& x(j+1)=Ax(j)+Bu(j),\\
& (x(j),u(j))\in\mathcal{Z},\quad j=0,\cdots,N-1\\
& (x_s,u_s)=M_\theta\theta,\\
& y_s=N_\theta\theta\\
& (x(N),\theta)\in\Omega_t^a
\end{aligned}$$

Considering the receding horizon policy, the control law is given by

$$\kappa_N(x)=u^0(0;x)$$

Since the set of constraints of $P_N(x)$ does not depend on y_t, its feasibility region does not depend on the target operating point y_t. Then there exists a polyhedral region $\mathcal{X}_N\subseteq X$ such that for all $x\in\mathcal{X}_N$, $P_N(x)$ is feasible. This is the set of initial states that can be admissibly steered to the projection of Ω_t^a onto x in N steps.

Consider the following assumption on the controller parameters:

Assumption 2.4

1. *Let $R\in\mathbb{R}^{m\times m}$ be a positive definite matrix and $Q\in\mathbb{R}^{n\times n}$ a positive semi-definite matrix such that the pair $(Q^{1/2},A)$ is observable.*

2. *Let $K\in\mathbb{R}^{m\times n}$ be a stabilizing control gain such that $(A+BK)$ has the eigenvalues in the unit circle.*

3. *Let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix such that:*

$$(A+BK)'P(A+BK)-P=-(Q+K'RK)$$

4. *Let $\Omega_t^a \subseteq \mathbb{R}^{n+m}$ be an admissible polyhedral invariant set for tracking for system* (2.1) *subject to* (2.2)*, for a given gain K. See 2.3.3 for more details.*

It can be considered that Ω_t^a contains the set of equilibrium points $\Omega_{eq} = \{(x_s,\theta) : (x_s,u_s) = M_\theta\theta, (x_s,u_s) \in \lambda\mathcal{Z}\}$. This is not restricting since if this is not the case, the convex hull of Ω_t^a and Ω_{eq} is also an invariant set for tracking that ensures this condition.

The set of admissible steady outputs consistent with the invariant set for tracking Ω_t^a is given by:

$$\{y_s = N_\theta\theta : (x_s,u_s) = M_\theta\theta, \text{and } (x_s,\theta) \in \Omega_t^a\}$$

This set is equal to the set of all admissible outputs for system (2.1) subject to (2.2), that is, $\mathcal{Y}_s$.

Taking into account the proposed conditions on the controller parameters, in the following theorem asymptotic stability and constraints satisfaction of the controlled system are proved.

Theorem 2.5 (Stability) *Consider that assumptions 2.1 and 2.4 hold and consider a given target operation point y_t. Then for any feasible initial state $x_0 \in \mathcal{X}_N$, the system controlled by the proposed MPC controller $\kappa_N(x)$ is stable, fulfils the constraints along the time and, besides*

(i) If $y_t \in \mathcal{Y}_s$ then the closed-loop system asymptotically converges to a steady state and input (x_t,u_t) such that $y_t = Cx_t + Du_t$.

(ii) In other case, the closed-loop system asymptotically converges to a steady state and input (x_s^*,u_s^*) and $y_s^* = Cx_s^* + Du_s^*$ where

$$y_s^* = \arg\min_{y_s\in\mathcal{Y}_s} V_O(y_s - y_t)$$

Proof: Consider that $x \in \mathcal{X}_N$ at time k, then the optimal cost function is given by $V_N^0(x) = V_N(x, \mathbf{u}^0(x), \theta^0(x))$, where $(\mathbf{u}^0(x), \theta^0(x))$ defines the optimal solution of $P_N(x)$ and $\mathbf{u}^0(x) = \{u^0(0;x), u^0(1;x), ..., u^0(N-1;x)\}$. The resultant optimal state sequence associated to $\mathbf{u}^0(x)$ is given by $\mathbf{x}^0(x) = \{x^0(0;x), x^0(1;x), ..., x^0(N-1;x), x^0(N;x)\}$, where $x^0(j;x) = \phi(j;x,\mathbf{u}^0(x))$ and $x^0(N;x) \in \Omega_t$.

As standard in MPC (Mayne et al., 2000; Rawlings and Mayne, 2009, Chapter 2), define the successor state at time $k+1$, $x^+ = Ax + Bu^0(0;x)$ and define also the following sequences:

$$\begin{aligned}\tilde{\mathbf{u}}(x) &\triangleq [u^0(1;x), \cdots, u^0(N-1;x), K(x^0(N;x) - x_s^0(x)) + u_s^0(x)]\\ \tilde{\theta}(x) &\triangleq \theta^0(x)\end{aligned}$$

where $(x_s^0, u_s^0) = M_\theta \theta^0$. Then, following a similar procedure to (Limon et al., 2008a), it is proved that $(\tilde{\mathbf{u}}(x), \tilde{\theta}(x))$ is a feasible solution for the optimization problem $P_N(x^+)$.

The state sequence due to $(\tilde{\mathbf{u}}(x), \tilde{\theta}(x))$ is $\tilde{\mathbf{x}} = \{x^0(1;x), x^0(2;x), ..., x^0(N;x), x^0(N+1;x)\}$, where $x^0(N+1;x) = (A+BK)x^0(N;x) + B(u_s^0(x) - Kx_s^0(x))$, which is clearly feasible. Compare now the optimal cost $V_N^0(x)$, with the cost given by $(\tilde{\mathbf{u}}(x), \tilde{\theta}(x))$, $\tilde{V}_N(x^+, \tilde{\mathbf{u}}(x), \tilde{\theta}(x))$. Taking into account the properties of the feasible nominal trajectories for x^+, the condition *(4)* of Assumption 2.4 and using standard procedures in MPC (Mayne et al., 2000; Rawlings and Mayne, 2009, Chapter 2) it is possible to obtain:

$$\begin{aligned}\tilde{V}_N(x^+;\tilde{\mathbf{u}},\tilde{\theta}) - V_N^0(x) = &-\|x - x_s^0(x)\|_Q^2 - \|u^0(0;x) - u_s^0(x)\|_R^2 - \|x^0(N) - x_s^0(x)\|_P^2 - V_O(y_s - y_t)\\ &+\|x^0(N;x) - x_s^0(x)\|_Q^2 + \|K(x^0(N;x) - x_s^0(x))\|_R^2\\ &+\|x^0(N+1;x) - x_s^0(x)\|_P^2 + V_O(y_s - y_t)\\ = &-\|x - x_s^0(x)\|_Q^2 - \|u^0(0;x) - u_s^0(x)\|_R^2\end{aligned}$$

By optimality, we have that $V_N^0(x^+) \le \tilde{V}_N(x^+;\tilde{\mathbf{u}},\tilde{\theta})$ and then:

$$V_N^0(x^+) - V_N^0(x) \quad \le \quad -\|x - x_s^0(x)\|_Q^2 - \|u^0(0;x) - u_s^0(x)\|_R^2$$

Taking into account that $V_N^0(x)$ is a positive definite convex function and that $V_N^0(x^+) - V_N^0(x) \le 0$, we have that (x_s^0, u_s^0) is an stable equilibrium point. Furthermore taking into account that $(Q^{1/2}, A)$ is observable, it is derived that

$$\lim_{k\to\infty}|x(k) - x_s^0(x(k))| = 0, \ \lim_{k\to\infty}|u(k) - u_s^0(x(k))| = 0$$

Hence the system converges to an operating point $(x_s^0, u_s^0) = M_\theta \theta^0$ such that $(x_s^0, \theta^0) \in \Omega_t^a$.

Now, it is proved that the system converges to an equilibrium point. Pick an $\varepsilon > 0$, then there exists a $k(\varepsilon)$ such that for all $k \geq k(\varepsilon)$, $|x - x_s^0(x)| < \varepsilon$ and $|u^0(0;x) - u_s^0(x)| < \varepsilon$. Then, removing the time dependence for the sake of simplicity, it is inferred that

$$\begin{aligned} |x^+ - x| &= |x^+ - x_s^0(x) + x_s^0(x) - x| \\ &\leq |x^+ - x_s^0(x)| + |x_s^0(x) - x| \\ &= |Ax + Bu^0(0;x) - Ax_s^0(x) - Bu_s^0(x)| + |x_s^0(x) - x| \\ &\leq |A - I||x - x_s^0(x)| + |B||u^0(0;x) - u_s^0(x)| \\ &\leq c\varepsilon \end{aligned}$$

where $c = |A - I| + |B|$. Therefore, for a given $\varepsilon > 0$, there exists a $k(\varepsilon)$ such that $|x^+ - x| \leq c\varepsilon$. Hence, the system converges to a steady state x_s^* and this is such that $x_s^* = x_s^0(x_s^*) = Ax_s^* + Bu_s^*$.

The proof will be finished demonstrating that the equilibrium point (x_s^*, u_s^*) is the minimizer of the offset cost function $V_O(y_s - y_t)$, proving the second assertion of the theorem. The first one is a direct consequence of the latter.

This result is obtained by contradiction. Consider the following set of the optimal solutions:

$$\Gamma = \{y_s : y_s = arg \min_{y \in \mathcal{Y}_s} V_O(y - y_t)\}$$

Consider that $y_s^* \notin \Gamma$. Then there exists a $\tilde{y}_s \in \Gamma$, such that $V_O(\tilde{y}_s - y_t) < V_O(y_s^* - y_t)$. Define $\tilde{\theta}$ as a parameter (contained in the projection of Ω_t^a onto θ) such that $\tilde{y}_s = N_\theta \tilde{\theta}$. In can be proved (Alvarado, 2007) that there exists a $\hat{\lambda} \in [0, 1)$ such that for every $\lambda \in [\hat{\lambda}, 1)$, the parameter $\hat{\theta} = \lambda\theta^* + (1 - \lambda)\tilde{\theta}$ is such that the control law $u = Kx + L\hat{\theta}$ (with $L = [-K, I_m]M_\theta$) steers the system from x_s^* to $\hat{x}_s$ fulfilling the constraints.

Defining as $\mathbf{u}$ the sequence of control actions derived from the control law $u = K(x - \hat{x}_s) + \hat{u}_s$, it is inferred that $(\mathbf{u}, \hat{\theta})$ is a feasible solution for $P_N(x_s^*)$ (Limon et al., 2008a). Then, from assumption 2.4,

$$\begin{aligned} V_N^0(x_s^*) &\leq V_N(x_s^*; \mathbf{u}, \hat{y}_s) \\ &= \sum_{i=0}^{N-1} \overbrace{\|x(i) - \hat{x}_s\|_Q^2 + \|K(x(i) - \hat{x}_s)\|_R^2}^{\|x(i)-\hat{x}_s\|^2_{(Q+K'RK)}} + \|x(N) - \hat{x}_s\|_P^2 + V_O(\hat{y}_s - y_t) \\ &= \|x_s^* - \hat{x}_s\|_P^2 + V_O(\hat{y}_s - y_t) \end{aligned}$$

Then, considering the previous statements:

$$\begin{aligned} V_N(x_s^*, y_t; \mathbf{u}, \hat{y}_s) &= \|x_s^* - \hat{x}_s\|_P^2 + V_O(\hat{y}_s - y_t) \\ &= \|\theta^* - \hat{\theta}\|_{M_x' P M_x}^2 + V_O(\hat{y}_s - y_t) \\ &= (1-\lambda)^2 \|\theta^* - \tilde{\theta}\|_{M_x' P M_x}^2 + V_O(\hat{y}_s - y_t) \end{aligned}$$

The partial of V_N about λ is:

$$\frac{\partial V_N}{\partial \lambda} \;=\; -2(1-\lambda)\|\theta^* - \tilde{\theta}\|_{M_x' P M_x}^2 + g'(y_s^* - \tilde{y}_s)$$

where $g' \in \partial V_O(\hat{y}_s - y_t)$, defining $\partial V_O(\hat{y}_s - y_t)$ as the subdifferential of $V_O(\hat{y}_s - y_t)$. Evaluating this partial for $\lambda = 1$ we obtain that:

$$\left.\frac{\partial V_N}{\partial \lambda}\right|_{\lambda=1} = g^{*'}(y_s^* - \tilde{y}_s)$$

where $g^{*T} \in \partial V_O(y_s^* - y_t)$, defining $\partial V_O(y_s^* - y_t)$ as the subdifferential of $V_O(y_s^* - y_t)$. Taking into account that V_O is a subdifferentiable function, we can state that

$$g^{*T}(y_s^* - \tilde{y}_s) \geq V_O(y_s^* - y_t) - V_O(\tilde{y}_s - y_t)$$

Considering that $V_O(y_s^* - y_t) - V_O(\tilde{y}_s - y_t) > 0$, it can be derived that

$$\left.\frac{\partial V_N}{\partial \lambda}\right|_{\lambda=1} \;\geq\; V_O(y_s^* - y_t) - V_O(\tilde{y}_s - y_t) > 0$$

This means that there exists a $\lambda \in [\hat{\lambda}, 1)$ such that $V_N(x_s^*; \mathbf{u}, \hat{y}_s)$ is smaller than the value of $V_N(x_s^*; \mathbf{u}, \hat{y}_s)$ for $\lambda = 1$, which equals to $V_N^0(x_s^*)$.

This contradicts the optimality of the solution and hence it is proved that (x_s^*, u_s^*) is the optimal steady state of the system.

Finally, the fact that (x_s^*, u_s^*) is a stable equilibrium point for the closed-loop system is proved. That is, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $|x(0) - x_s^*| \leq \delta$, then $|x(k) - x_s^*| \leq \varepsilon$. Notice that the region $\mathcal{B} = \{x : |x(k) - x_s^*| \leq \varepsilon\} \subseteq \mathcal{X}_N$ and this is true because $x_s^* \in int(\mathcal{X}_N)$.

Hence, define the function $W(x, y_t) = V_N^0(x, y_t) - V_O(y_s^* - y_t)$. Then, $W(x_s^*, y_t) = 0$. This function is such that $\alpha_W(|x - x_s^*|) \leq W(x, y_t) \leq \beta_W(|x - x_s^*|)$, where α_W and β_W are suitable $\mathcal{K}_\infty$ functions. In fact:

- $W(x, y_t) \geq \alpha_l(|x - x_s^0|) + \alpha_O(|x_s^0 - x_s^*|)$. This comes from the fact that the stage cost function is a positive definite function and from the definition of V_O. Then

$$\begin{aligned} W(x, y_t) &\geq \alpha_W(|x - x_s^0| + |x_s^0 - x_s^*|) \\ &\geq \alpha_W(|x - x_s^*|) \end{aligned}$$

- Notice that $V_N^0(x, y_t) \leq V_N(x, y_s^*) + V_O(y_s^* - y_t)$. Since $V_N^0(x, y_t)$ is continuous, there exists a $\mathcal{K}_\infty$ function β_W such that $V_N(x, y_s^*) \leq \beta_W(|x - x_s^*|)$. Hence $W(x, y_t) \leq \beta_W(|x - x_s^*|)$.

Then, $\alpha_W(|x(k) - x_s^*|) \leq W(x(k), y_t) \leq W(x(0), y_t) \leq \beta_W(|x - x_s^*|)$ and, hence, $|x(k) - x_s^*| \leq \alpha_W^{-1} \circ \beta_W(|x(0) - x_s^*|)$. So, picking $\delta = \beta_W^{-1} \circ \alpha_W(\varepsilon)$, then $|x(k) - x_s^*| \leq \alpha_W^{-1} \circ \beta_W(\delta) \leq \varepsilon$, proving the stability of x_s^*. ■

2.5 Properties of the proposed controller

2.5.1 Steady state optimization.

It is not unusual that the output target y_t is not contained in $\mathcal{Y}_s$. This may happen when there not exists an admissible operating point which steady output equals to the target or when the target is not a possible steady output of the system (that is, this is not in the subspace spanned by the columns of matrix N_θ). To deal with this situation in predictive controllers, the standard solution is to add an upper level steady state optimizer to decide the best reachable target of the controller (Rao and Rawlings, 1999).

From the latter theorem it can be clearly seen that in this case, the proposed controller steers the system to the optimal operating point according to the offset cost function $V_O(.)$. Then it can be considered that the proposed controller has a steady state optimizer built in and $V_O(.)$ defines the function to optimize. See that the only mild assumptions on this function are to be convex, positive definite, subdifferentiable and zero when the entry is null (to ensure offset-free control if $y_t \in \mathcal{Y}_s$).

2.5.2 Offset cost function and stability.

Taking into account theorem 2.5, stability is proved for any offset cost function satisfying assumption 2.4. Therefore, if this cost function varies with the time, the results of the theorem still hold.
This property allows as to tune the cost function along the time maintaining the stabilizing properties of the controller. Besides, this property can be exploited to consider an offset cost function which depends on the target, namely $V_O(y_t; y_s - y_t)$ defining different optimal criterium for the operating point selection depending on the chosen target (the example illustrates this idea).

2.5.3 Larger domain of attraction.

For a given prediction horizon, the proposed controller provides a larger region of attraction than the one of the MPC for regulation. This remarkable property allows to extend the controllability of the predictive controller to a larger region at expense of m additional decision variables. This increment of computational cost is similar to one the derived from incrementing the prediction horizon by 1. This property makes the proposed controller interesting even for regulation objectives.
On the other hand, for a given region of initial states, the necessary prediction horizon to control the system is potentially smaller, which implies a lower computational cost.

2.5.4 Robustness and output feedback

It has been demonstrated that asymptotically stabilizing predictive control laws may exhibit zero-robustness, that is, any disturbance may make the controller to be unfeasible or the asymptotic stability property may not hold (Grimm et al., 2004). In this case, taking into account that the control law is derived from a multiparametric convex problem, the closed-loop system is input-to-state stable for sufficiently small uncertainties (Limon et al., 2009a).
This property is very interesting for an output feedback formulation (Messina et al., 2005), since it allows to ensure asymptotic stability for the control law based on the estimated state using an asymptotically stable observer. A robust formulation of the proposed controller can be obtained by extending the formulation presented in (Alvarado et al., 2007b) for state feedback and (Alvarado et al., 2007a) for output feed-

back. In this case, offset free control can be achieved by means of disturbances models (Pannocchia and Kerrigan, 2005) or adding an outer loop which manages the targets (Alvarado, 2007).

2.5.5 QP formulation.

The optimization problem $P_N(x)$ is a convex mathematical programming problem that can be efficiently solved. In the case that the offset cost function $V_O(y_s - y_t)$ is such that the region $\{y_s : V_O(y_s - y_t) \leq 0\}$ is polyhedral, then $P_N(x)$ can be posed as a quadratic programming by means of an epigraph formulation.

2.5.6 Explicit formulation.

The structure of the equivalent optimization problem ensures that the proposed control law $\kappa_N(x)$ is a piecewise affine function of (x, y_t) that can by explicitly calculated by means of the existing multiparametric programming tools (Bemporad et al., 2002).

In the following section it is demonstrated that the proposed controller provides a locally optimal control law.

2.6 Local Optimality

In this section, the local optimality properties of the MPC controllers is presented, and how the MPC for tracking presented in this chapter is able to provide it.

Consider that system (2.1) is controlled by the control law $u = \kappa(x, y_t)$ to steer the system to the target $y_t \in \mathcal{Y}_s$. Let θ_t be the unique parameter such that $y_t = N_\theta \theta_t$ and let (x_t, u_t) be given by $(x_t, u_t) = M_\theta \theta_t$. Assume that matrix N_θ is full column rank. Consider also a quadratic cost function of the closed-loop system evolution when the initial state is x, given by

$$V_\infty(x, y_t, \kappa(\cdot, y_t)) = \sum_{j=0}^{\infty} \|x(j) - x_t\|_Q^2 + \|\kappa(x(j), y_t) - u_t\|_R^2$$

where $x(j) = \phi(j; x, \kappa(\cdot, y_t))$ is calculated from the recursion $x(i+1) = Ax(i) + B\kappa(x(i), y_t)$ for $i = 0, \cdots, j-1$ with $x(0) = x$. A control law $\kappa_\infty(x, y_t)$ is said to be optimal if it is admissible (namely, the constraints are fulfilled along the closed-loop evolution) and it is the one which minimizes the cost $V_\infty(x, y_t, \kappa(\cdot, y_t))$ for all admissible x. It is clear that the optimal control law (the so-called Linear Quadratic Regulator) is the best control law to be designed according to the given performance index. The optimal cost function is denoted as $V^0_\infty(x, y_t) = V_\infty(x, y_t, \kappa_\infty(\cdot, y_t))$. The calculation of the optimal control law $\kappa_\infty(x, y_t)$ may be computationally unaffordable for constrained systems, while for unconstrained, it can be obtained from the solution of a Riccati's equation.

Model predictive controllers can be considered as suboptimal controllers since the cost function is only minimized for a finite prediction horizon. The standard MPC control law to regulate the system to the target y_t, $\kappa^r_N(x, y_t)$, is derived from the following optimization problem $P^r_N(x, y_t)$

$$
\begin{aligned}
V_N^{r,0}(x, y_t) \;&=\; \min_{\mathbf{u},\theta} \sum_{j=0}^{N-1} \|x(j)-x_s\|^2_Q+\|u(j)-u_s\|^2_R + \|x(N)-x_s\|^2_P && \text{(2.8a)}\\
s.t.\;\; & x(0) = x, && \text{(2.8b)}\\
& x(j+1) = Ax(j) + Bu(j), && \text{(2.8c)}\\
& (x(j), u(j)) \in \mathcal{Z}, \quad j = 0, \cdots, N{-}1 && \text{(2.8d)}\\
& (x_s, u_s) = M_\theta \theta, && \text{(2.8e)}\\
& y_s = N_\theta \theta, && \text{(2.8f)}\\
& (x(N), \theta) \in \Omega^a_t && \text{(2.8g)}\\
& \|y_s - y_t\|_\infty = 0 && \text{(2.8h)}
\end{aligned}
$$

This optimization problem is feasible for any x in a polyhedral region denoted as $\mathcal{X}^r_N(y_t)$. Under certain assumptions (Mayne et al., 2000), for any feasible initial state $x \in \mathcal{X}^r_N(y_t)$, the control law $\kappa^r_N(x, y_t)$ steers the system to the target fulfilling the constraints. However, this control law is suboptimal in the sense that it does not minimizes $V_\infty(x, y_t, \kappa^r_N(\cdot, y_t))$. Fortunately, as stated in the following lemma, if the terminal cost function is the optimal cost of the unconstrained LQR, then the resulting finite horizon MPC is equal to the constrained LQR in a neighborhood of the terminal region (Hu and Linnemann, 2002; Bemporad et al., 2002).

Lemma 2.6 *Consider that assumptions 2.1 and 2.4 hold. Consider that the terminal control gain K is the one of the unconstrained linear quadratic regulator. Let θ_t be the* *parameter such that* $y_t = N_\theta \theta_t$*. Define the set* $\Upsilon_N(y_t) \subset \mathbb{R}^n$ *as*

$$\Upsilon_N(y_t) = \{\bar{x} \in \mathbb{R}^n : (\phi(N;\bar{x},\kappa_\infty(\cdot,y_t)),\theta_t) \in \Omega^w_{t,K}\}$$

Then for all $x \in \Upsilon_N(y_t)$*,* $V_N^{r,0}(x,y_t) = V_\infty^0(x,y_t)$ *and* $\kappa_N^r(x,y_t) = \kappa_\infty(x,y_t)$*.*

This lemma directly stems from (Hu and Linnemann, 2002, Thm. 2).

The proposed MPC for tracking might not ensure this local optimality property under assumptions of lemma 2.6 due to the artificial steady state and input and the functional cost to minimize. However, as it is demonstrated in the following property, under some conditions on the offset cost function $V_O(\cdot)$, this property holds.

Assumption 2.7 *Let the offset cost function* $V_O(.)$ *be defined as in 2.2 and such that*

$$\alpha\|y\| \leq V_O(y) \leq \beta\|y\|, \quad \forall y \in \mathcal{Y}_s$$

where α *and* β *are positive real constants.*

Lemma 2.8
Consider that assumptions 2.1, 2.4 and 2.7 hold. Then there exists a $\alpha^* > 0$ *such that for all* $\alpha \geq \alpha^*$*:*

- *The proposed MPC for tracking is equal to the MPC for regulation, that is* $\kappa_N(x,y_t) = \kappa_N^r(x,y_t)$ *and* $V_N^0(x,y_t) = V_N^{r,0}(x,y_t)$ *for all* $x \in \mathcal{X}_N^r(y_t)$*.*

- *If the terminal control gain* K *is the one of the unconstrained linear quadratic regulator, then the MPC for tracking control law* $\kappa_N(x,y_t)$ *is equal to the optimal control law* $\kappa_\infty(x,y_t)$ *for all* $x \in \Upsilon(y_t)$*.*

Proof: First, define the following optimization problem $P^m_{N,\alpha}(x, y_t; \alpha)$ as:

$$
\begin{aligned}
V^{m,0}_{N,\alpha}(x, y_t, \alpha) \; &= \; \min_{\mathbf{u},\theta} \sum_{j=0}^{N-1} \|x(j) - x_s\|^2_Q + \|u(j) - u_s\|^2_R + \|x(N) - x_s\|^2_P + \alpha \|y_s - y_t\|_1 \\
s.t. \quad & x(0) = x, \\
& x(j+1) = Ax(j) + Bu(j), \\
& (x(j), u(j)) \in \mathcal{Z}, \quad j = 0, \cdots, N{-}1 \\
& (x_s, u_s) = M_\theta \bar{\theta}, \\
& y_s = N_\theta \bar{\theta}, \\
& (x(N), \theta) \in \Omega^a_t
\end{aligned}
$$

This optimization problem $P^m_{N,\alpha}(x, y_t; \alpha)$ results from the optimization problem $P^r_N(x, y_t)$ with the last constraint posed as an exact penalty function (Luenberger, 1984). Therefore, there exists a finite constant $\alpha^* > 0$ such that for all $\alpha \geq \alpha^*$, $V^{m,0}_{N,\alpha}(x, y_t) = V^{r,0}_N(x, y_t)$ for all $x \in \mathcal{X}^r_N(y_t)$ (Luenberger, 1984; Boyd and Vandenberghe, 2006).

Considering that $V_O(y) \leq \beta \|y\|$. Then

$$V^{m,0}_{N,\alpha}(x, y_t) \leq V^0_N(x, y_t) \leq V^{m,0}_{N,\beta}(x, y_t)$$

Since $\beta \geq \alpha \geq \alpha^*$, we have that for all $x \in \mathcal{X}^r_N(y_t)$

$$V^{r,0}_N(x, y_t) \leq V^0_N(x, y_t) \leq V^{r,0}_N(x, y_t)$$

and hence $V^0_N(x, y_t) = V^{r,0}_N(x, y_t)$.

The second claim is derived from lemma 2.6 observing that $\Upsilon_N(y_t) \subseteq \mathcal{X}^r_N(y_t)$. ■

Remark 2.9 *In virtue of the well-known result on the exact penalty functions (Luenberger, 1984), the constant α can be chosen such that $\|\nu(x, y_t)\|_1 \leq \alpha$, where $\nu(x, y_t)$ is the Lagrange multiplier of the equality constraint $\|y_s - y_t\|_\infty = 0$ of the optimization problem $P^r_N(x, y_t)$. Since the optimization problem depends on the parameters (x, y_t), the value of this Lagrange multiplier also depends on (x, y_t).*

Remark 2.10 *The local optimality property can be ensured using any norm, thanks to the property of equivalence of the norms, that is $\exists c > 0$ such that $\|x\|_q \geq c\|x\|_1$. Otherwise, the square of a norm cannot be used. With the $\|.\|^2_q$ norm, in fact, there will be always a local optimality gap for a finite value of α since $\|.\|^2_q$ is a (not exact) penalty function, (Luenberger, 1984). That gap can be reduced by means of a suitable penalization of the offset cost function, (Alvarado, 2007).*

Remark 2.11 *Assumption 2.7 can be easily satisfied for any function $\hat{V}_O(.)$ considering as offset cost function $V_O(y) = \max(\hat{V}_O(y), \alpha\|y\|)$ which is a convex function. If $\mathcal{Y}_s$ is bounded, the upper bound condition is directly fulfilled.*

Some questions arise from this result as how a suitable value of the parameter α can be determined for all possible set of parameters. Another issue is if there exists a region where local optimality property holds for a given value of α. These issue are analyzed in the following section.

2.6.1 Characterization of the region of local optimality

From the previously presented results, it can be seen that this issue can be studied by characterizing the region where the norm of the Lagrange multiplier $\nu(x, y_t)$ is lower than or equal to α. Once this region is determined, the open questions on the local optimality can be answered. The characterization of this region is done by means of results of multiparametric quadratic programming problems (Bemporad et al., 2002; Jones and Maciejowski, 2006; Morari et al., 2008).

To this aim, firstly, notice that the optimization problem $P_N^r(x, y_t)$ is a multiparametric problem and the set of parameters (x, y_t) such that $P_N^r(x, y_t)$ is feasible is given by $\Gamma = \{(x, y_t) : x \in \mathcal{X}_N^r(y_t)\}$. It can be proved that this set is a polytope.

This optimization problem can be casted as a multiparametric quadratic programming (mp-QP) problem (Bemporad et al., 2002) in the set of the parameters $(x, y_t) \in \Gamma$, which can be defined as:

$$\begin{array}{ll} \min\limits_z & \frac{1}{2} z' H z \\ s.t. & Gz \leq W + S_1 x + S_2 y_t \\ & Fz = Y + T_1 x + T_2 y_t \end{array} \tag{2.9}$$

where

$$z = \begin{bmatrix} u \\ \theta \end{bmatrix} + J_1 x + J_2 y_t \tag{2.10}$$

with J_1 and J_2 suitable matrices. $Gz \leq W + S_1 x + S_2 y_t$ describes the restrictions (2.8b)-(2.8g), and $Fz = Y + T_1 x + T_2 y_t$ is the only equality constraint represented by equation (2.8h). Notice that $H > 0$, then the problem is strictly convex.

The Karush-Kuhn-Tucker (KKT) optimality conditions (Boyd and Vandenberghe, 2006) for this problem are given by:

$$Hz + G'\lambda + F'\nu = 0 \tag{2.11a}$$
$$\lambda(Gz - W - S_1 x - S_2 y_t) = 0 \tag{2.11b}$$
$$\lambda \geq 0 \tag{2.11c}$$
$$Gz - W - S_1 x - S_2 y_t \leq 0 \tag{2.11d}$$
$$Fz - Y - T_1 x - T_2 y_t = 0 \tag{2.11e}$$

Solving (2.11a) for z and substituting in the other equations, we obtain a new set of constraints for the Lagrange dual problem associated with the problem (2.9) which depends on (λ, ν, x, y_t). Then the following region:

$$\Delta = \left\{ (\lambda, \nu, x, y_t) : \;\middle|\; \begin{array}{l} \lambda'(GH^{-1}G'\lambda + GH^{-1}F'\nu + W + S_1 x + S_2 y_t) = 0 \\ \lambda \geq 0 \\ -(GH^{-1}G'\lambda + GH^{-1}F'\nu + W + S_1 x + S_2 y_t) \leq 0 \\ FH^{-1}G'\lambda + FH^{-1}F'\nu + Y + T_1 x + T_2 y_t = 0 \end{array} \right\} \tag{2.12}$$

defines the set of (λ, ν, x, y_t) which is solution of the KKT conditions. Thus, for any $(x, y_t) \in Proj_{(x,y_t)}\Delta$, the solution of the KKT equations is $(\lambda(x, y_t), \nu(x, y_t))$ such that

$$(x, y_t, \lambda(x, y_t), \nu(x, y_t)) \in \Delta$$

Notice that $Proj_{(x,y_t)}\Delta$ is the set of (x, y_t) where a feasible solution exists and hence $Proj_{(x,y_t)}\Delta = \Gamma$ and it is polytope (Boyd and Vandenberghe, 2006).

Following the same arguments of (Bemporad et al., 2002), the finite number of inequality constraints makes that there exists a finite combination of possible active constraints. Consider the j'th combination and assume that $\breve{\lambda}^j$ and $\tilde{\lambda}^j$ denote the Lagrange multipliers vectors set of inactive and active inequality constraints respectively. Let $\breve{G}^j$, $\breve{W}^j$, $\breve{S}_1^j$, $\breve{S}_2^j$, and $\tilde{G}^j$, $\tilde{W}^j$, $\tilde{S}_1^j$, $\tilde{S}_2^j$ be the corresponding matrices derived from a suitable partition of matrices G, W, S_1 and S_2 for the set of inactive and active constraints. In virtue of the complementary slackness condition, we have that $\breve{\lambda}^j = 0$ for inactive constraints and $\tilde{G}^j H^{-1}\tilde{G}'^j\tilde{\lambda}^j + \tilde{G}^j H^{-1}F'\nu + \tilde{W}^j + \tilde{S}_1^j x + \tilde{S}_2^j y_t = 0$ for active constraints. Then, the j'th combination of active constraints remains active for every

(x, y_t, λ, ν) contained in the following polyhedral region:

$$\Delta_j = \left\{ (\lambda,\nu,x,y_t) : \lambda = (\tilde{\lambda}^j, \breve{\lambda}^j) \;\middle|\; \begin{array}{l} \breve{\lambda}^j = 0 \\ \tilde{\lambda}^j \geq 0, \quad j = 1, ..., N \\ \breve{G}^j H^{-1} F' \nu + \breve{W}^j + \breve{S}_1{}^j x + \breve{S}_2{}^j y_t > 0 \\ \tilde{\lambda}^j = -(\tilde{G}^j H^{-1} \tilde{G}'^j)^{-1} (\tilde{G}^j H^{-1} F' \nu + \tilde{W}^j + \tilde{S}_1{}^j x + \tilde{S}_2{}^j y_t) \\ F H^{-1} \tilde{G}'^j \tilde{\lambda}^j + F H^{-1} F' \nu + Y + T_1 x + T_2 y_t = 0 \end{array} \right\} \tag{2.13}$$

It is clear that, the union of every region Δ_j of a possible combination of active constraints, is such that $\Delta = \bigcup_j \Delta_j$ and hence Δ is a polygon.

Using these results, the maximum and the minimum value of $\|\nu(x, y_t)\|_1$ for all possible values of (x, y_t) can be computed, that is, the values of α_{min} and α_{max} such that for all $(x, y_t) \in \Gamma$, $\alpha_{min} \leq \|\nu(x, y_t)\|_1 \leq \alpha_{max}$. These are calculated by solving the following optimization problems:

$$\alpha_{max} = \max_{(x,y_t,\lambda,\nu)\in\Delta} \|\nu\|_1 = \max_j \left(\sup_{(x,y_t,\lambda,\nu)\in\Delta_j} \|\nu\|_1 \right) \tag{2.14}$$

$$\alpha_{min} = \min_{(x,y_t,\lambda,\nu)\in\Delta} \|\nu\|_1 = \min_j \left(\inf_{(x,y_t,\lambda,\nu)\in\Delta_j} \|\nu\|_1 \right) \tag{2.15}$$

It is remarkable that each supremum and infimum can be calculated by solving a set of linear programming (LP) problems in the closure of Δ_j. Besides, since the optimization problem $P_N^r(x, y_t)$ is such that the solution of the KKT conditions is unique, then the value of α_{max} is finite.

We are also interested in characterizing the set of (x, y_t), $\Gamma(\alpha)$, such that the norm of the associate Lagrange multiplier $\nu(x, y_t)$ is bounded by α, that is:

$$\Gamma(\alpha) = \{(x, y_t) : \exists(\lambda, \nu) \text{ s.t. } (\lambda, \nu, x, y_t) \in \Delta \text{ and } \|\nu\|_1 \leq \alpha\}$$

This region can be characterized by means of the polyhedral partition of Δ. Defining the set $\Gamma_j(\alpha) = \{(x, y_t) : \exists(\lambda, \nu) \text{ s.t. } (\lambda, \nu, x, y_t) \in \Delta_j \text{ and } \|\nu\|_1 \leq \alpha\}$, which is a polyhedron, it can be seen that $\Gamma(\alpha)$ is a polygon given by $\Gamma(\alpha) = \bigcup_j \Gamma_j(\alpha)$. Notice that set $\Gamma(\alpha)$ is non-empty for $\alpha > \alpha_{min}$. Moreover if $\alpha_{min} < \alpha_a \leq \alpha_b$, then for all $(x, y_t) \in \Gamma(\alpha_a)$, $\|\nu(x, y_t)\|_1 \leq \alpha_a \leq \alpha_b$ and hence $(x, y_t) \in \Gamma(\alpha_b)$. Therefore, $\Gamma(\alpha_a) \subseteq \Gamma(\alpha_b)$.

Resorting on the previously presented results, the following lemma can be derived.

Lemma 2.12
Consider that lemma 2.8 holds. Let α_{max} and α_{min} be the solution of (2.14) *and* (2.15) *respectively, then:*

- *For all $\alpha > \alpha_{min}$, there exists a polygon $\Gamma(\alpha)$ such that if $(x, y_t) \in \Gamma(\alpha)$, then $V_N^r(x, y_t) = V_N(x, y_t)$.*
- *For all $\alpha_{min} < \alpha_a \leq \alpha_b$, $\Gamma(\alpha_a) \subseteq \Gamma(\alpha_b)$. That is, $\Gamma(\alpha)$ grows monotonically with α.*
- *For all $\alpha \geq \alpha_{max}$, $\Gamma(\alpha) = Proj_{(x,y_t)}\Delta = \Gamma$.*

In the following theorem, the property of local optimality for the MPC for tracking is stated.

Theorem 2.13 (Local optimality)
Consider that lemma 2.8 and lemma 2.12 hold. Define the following region

$$\mathcal{W}(\alpha, y_t) = \{x \in \Upsilon_N(y_t) : (\phi(i; x, \kappa_N(\cdot, y_t)), y_t) \in \Gamma(\alpha), \forall i \geq 0\}$$

and let the terminal control gain K be the one of the unconstrained LQR. Then:

1. *For all $\alpha > \alpha_{min}$, $\mathcal{W}(\alpha, y_t)$ is a non-empty polygon and it is a positively invariant set of the controlled system.*
2. *If $\alpha_{min} < \alpha_a \leq \alpha_b$, then $\mathcal{W}(\alpha_a, y_t) \subseteq \mathcal{W}(\alpha_b, y_t)$.*
3. *If $\alpha > \alpha_{min}$, $x(0)$ and y_t are such that $x(0) \in \mathcal{X}_N^r(y_t)$, then*
   - *(a) There exists an instant $\bar{k}$ such that $x(\bar{k}) \in \mathcal{W}(\alpha, y_t)$ and $\kappa_N(x(k), y_t) = \kappa_\infty(x(k), y_t)$, for all $k \geq \bar{k}$.*
   - *(b) If $\alpha \geq \alpha_{max}$ then $\kappa_N(x(k), y_t) = \kappa_N^r(x(k), y_t)$ for all $k \geq 0$ and there exist an instant $\bar{k}$ such that $x(\bar{k}) \in \Upsilon_N(y_t)$ and $\kappa_N(x(k), y_t) = \kappa_\infty(x(k), y_t)$ for all $k \geq \bar{k}$.*

Proof:

- From lemma 2.6 we have that set $\Upsilon_N(y_t)$ is an invariant set for the system controlled by $u = \kappa_N^r(x, y_t)$ and besides, $\kappa_N^r(x, y_t) = \kappa_\infty(x, y_t)$. Since the control law $\kappa_N^r(x, y_t)$ is a piece-wise affine (PWA) function of (x, y_t), the controlled system is PWA and the region $\Upsilon_N(y_t)$ is a polygon (Kerrigan, 2000).

  On the other hand, set $\Xi(\alpha, y_t) = \{x : (\phi(i; x, \kappa_N(\cdot, y_t)), y_t) \in \Gamma(\alpha), \forall i \geq 0\}$ is the maximum invariant set for the controlled system contained in the set $\{x : (x, y_t) \in \Gamma(\alpha)\}$ and besides in virtue of lemma 2.12 for all $x \in \Xi(\alpha, y_t)$, $\kappa_N(x, y_t) = \kappa_N^r(x, y_t)$. The PWA nature of the control law ensures that $\Xi(\alpha, y_t)$ is a polygon.

  Finally, noticing that $\mathcal{W}(\alpha, y_t) = \Upsilon_N(y_t) \cap \Xi(\alpha, y_t)$, we infer that $\mathcal{W}(\alpha, y_t)$ is a positively invariant polygonal set for the system controlled by $\kappa_N(x, y_t)$ and for all $x \in \mathcal{W}(\alpha, y_t)$, $\kappa_N(x, y_t) = \kappa_N^r(x, y_t) = \kappa_\infty(x, y_t)$.

- Since $\alpha_a \leq \alpha_b$, $\Gamma(\alpha_a) \subseteq \Gamma(\alpha_b)$. In virtue of the monotonicity of the maximal invariant set, $\Xi(\alpha_a, y_t) \subseteq \Xi(\alpha_b, y_t)$ and this imply that $\mathcal{W}(\alpha_a, y_t) \subseteq \mathcal{W}(\alpha_b, y_t)$.

- If $x(0) \in \mathcal{X}_N^r(y_t)$, then the closed-loop system is asymptotically stable to $(x_s, u_s) = M_\theta \theta$, where θ is such that $y_t = N_\theta \theta$. Given that $\mathcal{W}(\alpha, y_t)$ has a non-empty interior and $x_s \in \mathcal{W}(\alpha, y_t)$ for any $\alpha > \alpha_{min}$, there exist a $\bar{k}$ when $x(\bar{k}) \in \mathcal{W}(\alpha, y_t)$. Due to the invariance of $\mathcal{W}(\alpha, y_t)$, $x(k) \in \mathcal{W}(\alpha, y_t)$ for all $k \geq \bar{k}$. Taking into account lemmas 2.8 and 2.12, $k_N(x(k), y_t) = k_\infty(x(k), y_t)$.

- From lemma 2.12, for all $\alpha \geq \alpha_{max}$ $\Gamma(\alpha) = \Gamma$, $\Xi(\alpha, y_t) = \mathcal{X}_N^r(y_t)$ and then $\mathcal{W}(\alpha, y_t) = \Upsilon_N(y_t)$. The result is derived from the last proposition.

■

From this theorem it can be inferred that for every $\alpha \geq \alpha_{min}$, the MPC for tracking is locally optimal in a certain region. In particular the value of α_{min} is interesting from a theoretical point of view, because it is the critical value from which there exists a region of local optimality. In order to ensure the local optimality property of the standard MPC, one would like to know the maximal region into which the local optimality applies. This region is given for any $\alpha \geq \alpha_{max}$. Then, from a practical point of view it is interesting to know α_{max}, but this requires the calculation of the partition of the feasibility region of the mp-QP and the solution of a number of LPs. In the following corollary it is proposed a method to calculate a value of $\alpha \geq \alpha_{min}$ for which the local optimality region is the invariant set for tracking, by means of a single LP.

Corollary 2.14 *Consider that hypotheses of theorem 2.13 hold. Let α_Ω be the solution* *of the following LP optimization problem:*

$$\begin{aligned} \alpha_\Omega &= \max_{x,\theta} \|(FH^{-1}F')^{-1}(Y + T_1x + T_2y_t)\|_1 \\ s.t. &\quad y_t = N_\theta\theta \\ &\quad (x,\theta) \in \Omega_t^a \end{aligned} \tag{2.16}$$

Assume that $\alpha \geq \alpha_\Omega$*, then for all* $x(0) \in \mathcal{X}_N^r(y_t)$*, there exists an instant* $\bar{k}$ *such that* $V_N^0(x(k), y_t) = V_\infty^0(x(k), y_t)$ *and* $\kappa_N(x(k), y_t) = \kappa_\infty(x(k), y_t)$*, for all* $k \geq \bar{k}$*.*

Proof: Assume that no inequality constraint is active, then the Lagrange multiplier λ is zero. In this case, the KKT conditions are

$$\begin{aligned} -GH^{-1}F'\nu - W - S_1x - S_2y_t &< 0 \\ -FH^{-1}F'\nu - Y - T_1x - T_2y_t &= 0 \end{aligned}$$

For any $(x, \theta) \in int(\Omega_t^a)$, the optimal control law is the one of the unconstrained LQR, that is $u = K_{LQR}(x - x_s) + u_s$, where $(x_s, u_s) = M_\theta\theta$ and $y_t = N_\theta\theta$, such that $(x, u) \in int(\mathcal{Z})$. This means that no inequality constraint is active. Considering that $u = K_{LQR}(x - x_s) + u_s$ is the optimal control law of the unconstrained LQR, then for any $(x, \theta) \in int(\Omega_t^a)$, $k_N(x, y_t) = K_{LQR}(x - x_s) + u_s$ and $x \in \Upsilon_N(y_t)$. Furthermore, for any $(x(\bar{k}), \theta) \in int(\Omega_t^a)$, $(x(k), \theta) \in int(\Omega_t^a)$ for any $\bar{k} \geq k$.
Hence, for any $(x, \theta) \in int(\Omega_t^a)$, $\lambda(x, y_t) = 0$, and then $\nu(x, y_t) = -[(FH^{-1}F')^{-1}(Y + T_1x + T_2y_t)]$. Moreover, $\|\nu(x, y_t)\|_1 \leq \alpha_\Omega$, for any $(x, \theta) \in int(\Omega_t^a)$.
Taking into account all this facts, if $\alpha \geq \alpha_\Omega$, then for any $x(0) \in \mathcal{X}_N$, there exists a $\bar{k} > 0$ such that $(x(\bar{k}), \theta) \in int(\Omega_t^a)$, and hence $k_N(x, y_t)$ is the optimal control law.

■

2.7 Illustrative example

In this example, the properties of the controller presented in this chapter, are proved in simulation. The system considered is the four tanks process, introduced in the Appendix A

2.7.0.1 Offset Minimization

The aim of the first test is to show the property of offset minimization of the controller. The offset cost function has been chosen as $V_O = \alpha\|y_s - y_t\|_\infty$. In the test, five references have been considered:$y_{t,1} = (0.3, 0.3)$, $y_{t,2} = (1.25, 1.25)$, $y_{t,3} = (0.35, 0.8)$, $y_{t,4} = (1, 0.8)$ and $y_{t,5} = (h_1^0, h_2^0)$. Notice that $y_{t,3}$ is not an equilibrium output for the system. The initial state is $x_0 = (0.65, 0.65, 0.6658, 0.6242)$. An MPC with $N = 3$ has been considered. The weighting matrices have been chosen as $Q = I_4$ and $R = 0.01 \times I_2$. Matrix P is the solution of the Riccati equation and $\alpha = 50$.

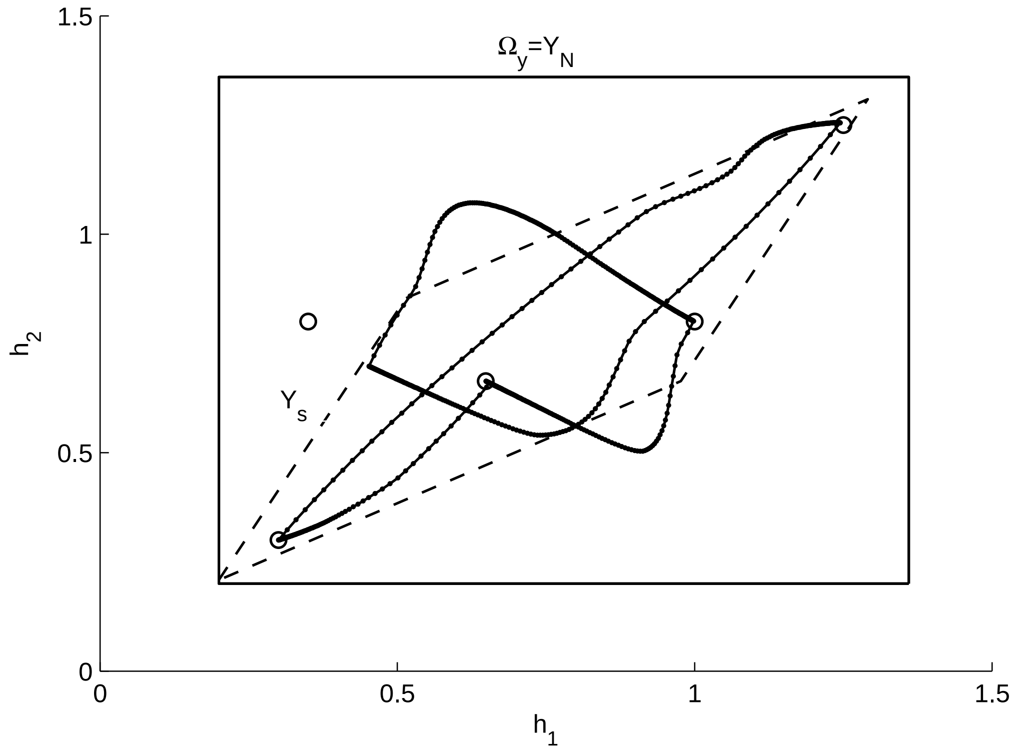


Figure 2.1: State-space and time evolutions.

The projection of the maximal invariant set for tracking onto y, Ω_y, the projection of the region of attraction onto y, $\mathcal{Y}_3$, the set of equilibrium levels $\mathcal{Y}_s$ and the state-space evolution of the levels h_1 and h_2 are shown in figure 2.1. The time evolutions are shown in figures 2.2 and 2.3. The reference is depicted in dashed-dotted line, while the artificial reference and the real evolution of the system are depicted respectively in dashed and solid line. As it can be seen, when the reference is an admissible setpoint, the system can reach it without any offset. When the reference changes to an unreachable setpoint,

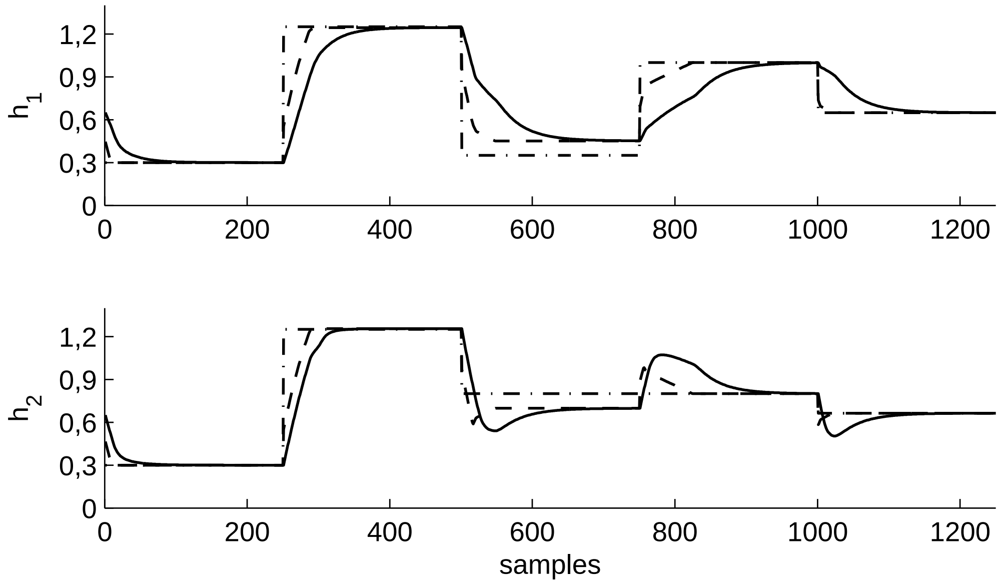


Figure 2.2: Evolution of the levels h_1 and h_2.

the controller leads the system to the closest equilibrium point, in the sense that the offset cost function is minimized.

2.7.0.2 Local Optimality

To illustrate the property of local optimality, the proposed controller has been compared with the MPC for tracking with quadratic offset cost function proposed in (Limon et al., 2008a). The difference of the optimal cost value of these two controllers, V_N^0 and $V_0^{q,0}$, with the one of the MPC for regulation, $V_0^{r,0}$ has been compared. To this aim, the quadratic offset cost function has been chosen as $\|y_s - y_t\|_{T_p}^2$ with $T_p = \alpha I_4$. The optimal MPC for tracking offset cost function has been chosen as a 1-norm, $V_O = \alpha\|y_s - y_t\|_1$. The system has been considered to be steered to the point $y = (h_1^0, h_2^0)$, with initial condition $y_0 = (1.25, 1.25)$. In figure 2.4 the value $V_N^{r,0} - V_N^0$ versus α is plotted in solid line and the value of $V_N^{r,0} - V_N^{q,0}$ versus α in dashed line. As it can be seen, $V_N^{r,0} - V_N^{q,0}$ tends to zero asymptotically while $V_N^{r,0} - V_N^0$ drops to (practically) zero for a certain value of α. This result shows that the optimality gap can be made arbitrarily small by means of a suitable penalization of the square of the 2 norm, and this value asymptotically converge to zero (Alvarado, 2007), while in the case of the 1-norm, the difference between the optimal value of the MPC for tracking cost function and the

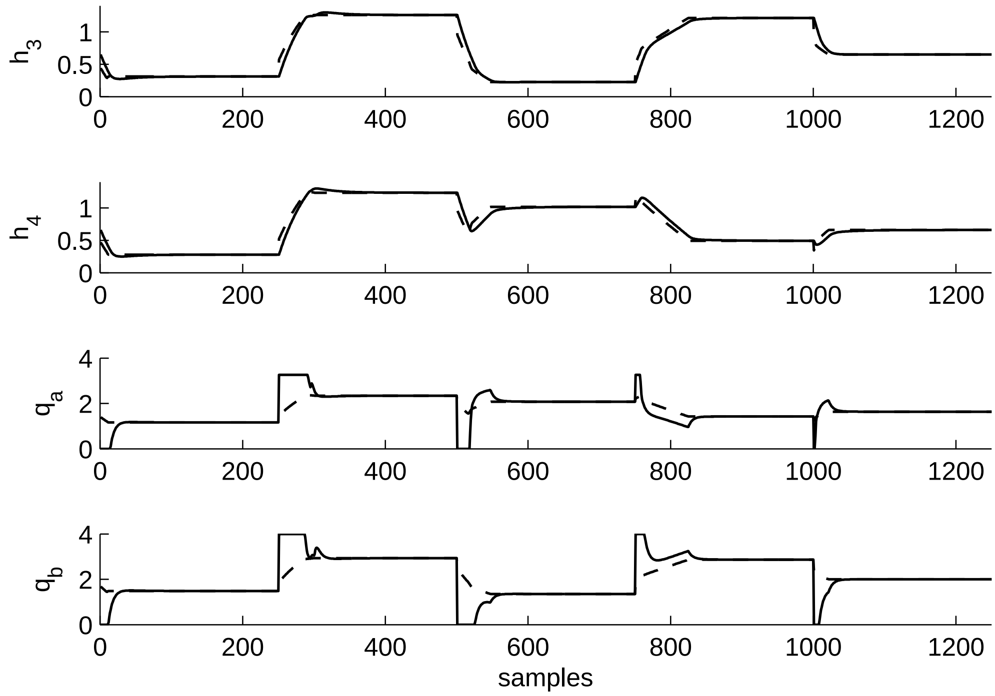
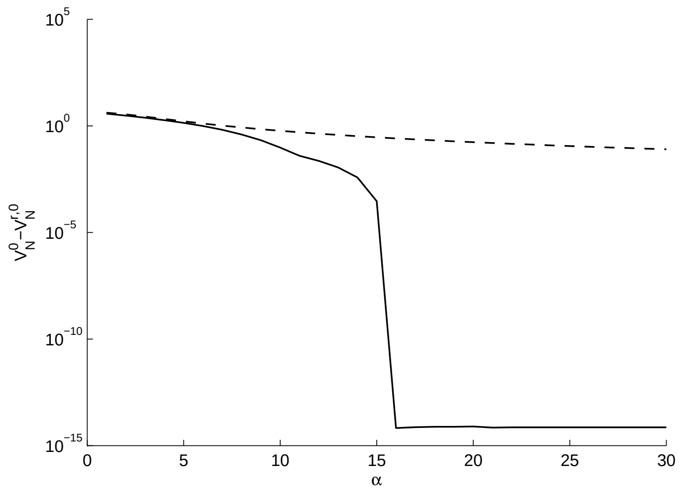


Figure 2.3: Evolution of the levels h_3 and h_4 and flows q_a and q_b.

standard MPC for regulation cost function becomes zero. This shows the benefit of the new formulation of the MPC for tracking.
Note how the value of $V_N^{r,0} - V_N^0$ drops to practically zero when $\alpha = 16$. As we said in section 2.6, this happens because the value of α becomes greater than the value of the Lagrange multiplier of the equality constraint of the regulation problem $P_{N,\alpha}^m(x, y_t; \alpha)$. In this test, the equality constraint of $P_{N,\alpha}^m(x, y_t; \alpha)$ has been chosen as an ∞-norm, and hence, to obtain an exact penalty function, the offset cost function of problem $P_N(x, y_t)$ has been chosen as a 1-norm. To point out this fact, consider that, for this example, the maximum value of the Lagrange multipliers of the equality constraint of the regulation problem $P_{N,\alpha}^m(x, y_t; \alpha)$, is $\alpha_{max} = 15.3868$. The value of α_Ω, calculated by solving problem (2.16), is $\alpha_\Omega = 14.6588$. In the table, the value of $V_N^{r,0} - V_N^0$ in case of different values of the parameter α is presented. Note how the value seriously decrease when α becomes equals to α_{max}. So, using the procedure described in section 2.6, we can determine the value of α_{max} such that $V_N^0(x, y_t) = V_N^{r,0}(x, y_t)$.

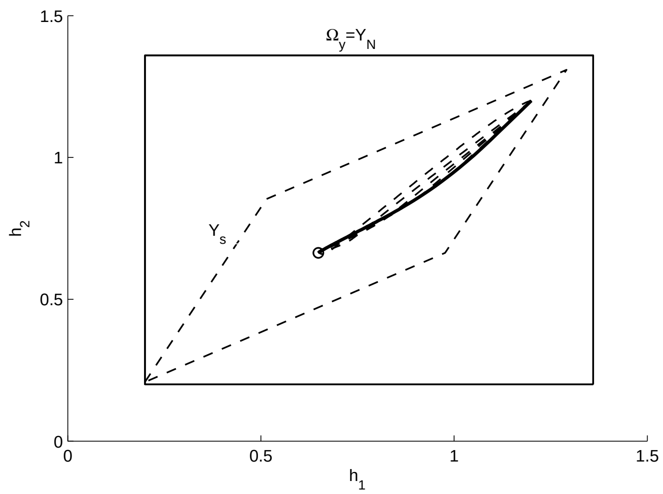
To definitely prove the optimal performances ensured by the proposed controller, the



α	$V_N^{r,0} - V_N^0$
14	0.0038
α_Ω	0.0010
15	$2.93e-4$
15.3	$1.47e-5$
15.38	$1.47e-5$
15.386	$1.15e-9$
α_{max}	$6.83e-15$
16	$6.83e-15$

Figure 2.4: Difference between the regulation cost and the tracking cost versus α.

optimal trajectories from the point $y_0 = (1.25, 1.25)$ to the point $y = (h_1^0, h_2^0)$ have been calculated, for a value of α that varies in the set $\alpha = \{2, 4, 6, 8, 10, 12, 14, \alpha_{max}, 18, 20\}$. In figure 2.5 the state-space trajectories and the values of the optimal cost V_∞ for α increasing are shown. See how the trajectories get better and how the value of the optimal cost decreases as the value of α increases. The optimal trajectory, in solid line, is the one for which $\alpha = \alpha_{max}$. Notice that value of the optimal cost decreases from $V_\infty = 84.2693$ to $V_\infty = 8.6084$ when α reaches the value of α_{max}.



α	V_∞
2	84.2693
4	40.6147
6	22.4751
8	13.8412
10	10.0638
12	8.9572
14	8.6989
15.3868	8.6084
18	8.6084
20	8.6084

Figure 2.5: State-space trajectories and optimal cost for α varying.

2.8 Conclusions

In this chapter an enhanced formulation of the MPC for tracking is presented. This formulation generalizes the original one by considering a general convex function as offset cost. This offset cost function allows to consider as target operating points states and inputs not consistent with the prediction model. This case is particularly interesting for non-square plants or for instance, when the target calculated by means of a non-linear model.

Under some assumptions, it is proved that the proposed controller steers the system to the target if this is admissible. If not, the controller converges to an admissible steady state optimum according to the offset cost function. Besides, the closed-loop evolution is also optimal in the sense that provides the best possible performance index.

CHAPTER 3

MPC for tracking target sets

3.1 Introduction

In this chapter the so called *zone control* problems and the application of MPC for tracking to this kind of problems, is presented. In particular, the concept of distance to a set is introduced and exploited for the design of the MPC control law.

3.1.1 Set-interval control in processing plants

In modern processing plants, MPC controllers are usually implemented as part of a multilevel hierarchy of control functions (Kassmann et al., 2000; Tatjewski, 2008). At the intermediary levels of this control structure, the process unit optimizer computes an optimal economic steady state and passes this information to the MPC in a lower level for implementation. The role of the MPC is then to drive the plant to the most profitable operating condition, fulfilling the constraints and minimizing the dynamic error along the path. In many cases, however, the optimal economic steady state operating condition is not given by a point in the output space (fixed set-point), but is a region into which the output should lie most of the time. In general, based on operational requirements, process outputs can be classified into two broad categories: 1) set-point controlled, outputs to be controlled at a desired value, and 2) set-interval controlled, outputs to be controlled within a desired range. For instance, production rate and product quality may fall into the first category, whereas process variables, such as level, pressure, and temperature in different units/streams may fall into the second category. The reasons for using set-interval control in real applications may be several, and they are all related to the process degrees of freedom: 1) In some problems some inputs of a square system without degrees of freedom are desired to be steered to a specific steady state values (input set-points), and then to account for the lack of degrees of freedom, the use of output zone control arises naturally (for example, it could be desirable, by economic reasons, to drive feed-rate to its maximum). 2) In another class of problems, there are highly correlated outputs to be controlled, and there are not enough inputs to control them independently. Controlling the correlated outputs within zones or ranges is one solution for this kind of problem (for instance, controlling the dense and dilute phase temperatures on an FCC regenerator). 3) A third important class of zone control problems relates to using the surge capacity of tanks to smooth out the operation of a unit. In this case, it is desirable to let the level of the tank float between limits, as necessary, to buffer disturbances between sections of a plant. Conceptually, the output intervals are not output constraints, since they are steady state desired zones that can be transitorily disregarded, while the (dynamic) constraints must be respected at each time. In addition, the determination of the output intervals is related to the steady state operability of the process, and it is not a trivial problem. A special care should be taken about the compatibility between the available input set (given by the input constraints) and the desired output set (given by the output intervals). In (Vinson and Georgakis, 2000) and (Lima and Georgakis, 2008), for instance, an operability index that quantify how much of the region of the desired outputs can be achieved using the available inputs, taking into account the expected disturbance set, is defined. As a result a methodology to obtain the tightest possible operable set of achievable output steady state is derived. Then, the operating control intervals should be subsets of these tightest intervals. In practice, however, the operators are not usually aware of these maximum zones and may select control zones that are not fully consistent with the maximum zones and the operating control zones may be fully or partly unreachable. The MPC controller has to be robust to this poor selection of the control zones.

3.1.2 Review of MPC controllers for set-interval control

Set-interval control has been accounted in many controllers in literature. In (Qin and Badgwell, 2003), it is mentioned that industrial controllers always provide a zone control option and two ways are proposed to implement zone control: i) defining upper and lower soft constraints; ii) using the set-point approximation of soft constraints to implement the upper and lower zone boundaries (the DMC-plus algorithm). The drawback of these industrial controllers is the lack of nominal stability. Another example of zone is presented in (Zanin et al., 2002). The great problem of the proposed strategy is that stability cannot be proved, even if an infinite horizon is used, since the control system keeps switching from one controller to another throughout the continuous operation of the process. A third example can be found in (Gonzalez and Odloak, 2009), where a closed loop stable MPC controller is presented. In this approach, the authors develop a controller that considers the zone control of the system outputs and incorporates steady state economic targets in the control cost function. The standard stability proofs are extended to the zone control strategy by considering the output set-points as additional decision variables of the control problem. Furthermore, a set of slack variables is included into the formulation to assure both, recursive feasibility of the on-line optimization problem and convergence of the system inputs to the targets. An extension of this strategy to the robust case, considering multi-model uncertainty, was proposed in (González et al., 2009).

From a theoretic point of view, the control objective of the zone control problem can be seen as a target set (in the output space) instead of a target point, since inside the zones there are no preferences between one point and another. In what follows, the controller presented in chapter 2, is extended to deal with the zone control, generalizing the conditions of the offset cost function to use a distance to a convex target set. This controller ensures recursive feasibility and convergence to the target set for any stabilizable plant. This property holds for any class of convex target sets and also in the case of time-varying target sets. For the case of polyhedral target sets, several formulations of the controller are proposed that allows to derive the control law from the solution of a single quadratic programming problem. One of these formulations allows also to consider target points and target sets simultaneously in such a way that the controller steers the plant to the target point if reachable while it steers the plant to the target set in the other case. Finally, it is worth to remark that the proposed controller inherits the properties of the controller proposed in chapter 2.

3.2 Problem Statement

Let a discrete-time linear system be described by:

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the current state of the system, $u \in \mathbb{R}^m$ is the current input, $y \in \mathbb{R}^p$ is the controlled output and x^+ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u})$, $j \in \mathbb{I}_{\geq 0}$, where $x = \phi(0; x, \mathbf{u})$. Note that no assumption is considered on the dimension

of the states, inputs and outputs and hence non square systems (namely $p > m$ or $p < m$) might be considered.
The controlled output is the variable used to define the target to be tracked by the controller. Since no assumption is made on matrices C and D, the outputs might be (a linear combination of) the states, (a linear combination of) the inputs or (a linear combination of) both.
The state of the system and the control input applied at sampling time k is denoted as $x(k)$ and $u(k)$ respectively. The system is subject to hard constraints on state and control:

$$(x(k), u(k)) \in \mathcal{Z} \tag{3.2}$$

for all $k \geq 0$. $\mathcal{Z} \subset \mathbb{R}^{n+m}$ is a compact convex polyhedron containing the origin in its interior.

Assumption 3.1 *The pair (A,B) is controllable and the state is measured at each sampling time.*

The problem we consider is the design of an MPC controller $\kappa_N^Z(x, \Gamma_t)$ such that for a given (possibly time varying) convex target set (zone region) Γ_t it steers the outputs of system to a steady value contained into the target region satisfying the constraints $(x(k), \kappa_N^Z(x(k), \Gamma_t)) \in \mathcal{Z}$ throughout its evolution.

3.3 MPC for tracking zone regions

In what follows, an extension of the MPC for tracking (Limon et al., 2008a; Ferramosca et al., 2009a) to the case of target sets is presented. In particular, in (Ferramosca et al., 2009a) the controller is formulated considering a generalized offset cost function. In this chapter, this controller is extended to the case of considering a zone control strategy. The control object is hence not to steer the system to a desired setpoint, but to lead the output into a specified region. To this aim, consider that the output target is a set, for instance a given polyhedron Γ_t. The cost function of the MPC proposed is, hence, given by:

$$V_N^Z(x, \Gamma_t; \mathbf{u}, \theta) \triangleq \sum_{j=0}^{N-1} \|x(j) - x_s\|_Q^2 + \|u(j) - u_s\|_R^2 + \|x(N) - x_s\|_P^2 + V_O(y_s, \Gamma_t) \tag{3.3}$$

where $x(j)$ denotes the prediction of the state j-samples ahead, the pair $(x_s, u_s) = M_\theta\theta$ is the artificial steady state and input and $y_s = N_\theta\theta$ the artificial output, all of them parameterized by θ; Γ_t is the zone in which the controlled variables have to be steered. The offset cost function $V_O(y_s, \Gamma_t)$ is such that the following assumption is ensured.

Assumption 3.2

1. *Γ_t is a compact convex set.*
2. *$V_O(y_s, \Gamma_t)$ is subdifferential and convex w.r.t. y_s.*
3. *If $y_s \in \Gamma_t$, then $V_O(y_s, \Gamma_t) \geq 0$. Otherwise, $V_O(y_s, \Gamma_t) > 0$.*

Let $P_N^Z(x, \Gamma_t)$ be the optimization problem that defines the controller for tracking of the zone region for the system constrained by $\mathcal{Z}$, with an horizon of length N and whose parameters are the actual state x and the target set Γ_t. This problem is defined as follows:

$$
\begin{aligned}
V_N^{Z,0}(x, \Gamma_t) \quad = \quad & \min_{\mathbf{u},\theta} V_N^Z(x, \Gamma_t; \mathbf{u}, \theta) && (3.4a)\\
s.t. \quad & x(0) = x, && (3.4b)\\
& x(j+1) = Ax(j) + Bu(j), && (3.4c)\\
& (x(j), u(j)) \in \mathcal{Z}, \quad j=0, \cdots, N-1 && (3.4d)\\
& (x_s, u_s) = M_\theta\theta, && (3.4e)\\
& y_s = N_\theta\theta && (3.4f)\\
& (x(N), \theta) \in \Omega_t^a && (3.4g)
\end{aligned}
$$

where Ω_t^a is the polyhedron that corresponds to the invariant set for tracking, with feedback controller K in the augmented state (x, θ). In what follows, the superscript ^{0} will denote the optimal solutions of the optimization problem.

Considering the receding horizon policy, the control law is given by

$$\kappa_N^Z(x, \Gamma_t) \triangleq u^0(0; x, \Gamma_t)$$

where $u^0(0; x, \Gamma_t)$ is the first element of the control sequence $\mathbf{u}^0(x, \Gamma_t)$ which is the optimal solution of problem $P_N^Z(x, \Gamma_t)$. Since the set of constraints of $P_N^Z(x, \Gamma_t)$ does not depend on Γ_t, its feasibility region does not depend on the target region Γ_t. Then there exists a polyhedral region $\mathcal{X}_N \subseteq \mathbb{R}^n$ such that for all $x \in \mathcal{X}_N$, $P_N^Z(x, \Gamma_t)$ is feasible. This is the set of initial states that can be admissibly steered in N steps to the projection of Ω_t^a onto x.

Consider the following assumption on the controller parameters:

Assumption 3.3

1. *Let $R \in \mathbb{R}^{m\times m}$ be a positive semidefinite matrix and $Q \in \mathbb{R}^{n\times n}$ a positive semi-definite matrix such that the pair $(Q^{1/2}, A)$ is observable.*
2. *Let $K \in \mathbb{R}^{m\times n}$ be a stabilizing control gain such that $(A+BK)$ is Hurwitz.*
3. *Let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix such that:*
$$(A+BK)'P(A+BK)-P=-(Q+K'RK)$$
4. *Let $\Omega_t^a \subseteq \mathbb{R}^{n+m}$ be an admissible polyhedral invariant set for tracking for system* (3.1) *subject to* (3.2)*, for a given gain K. See chapter 2 for more details.*

The set of admissible steady outputs consistent with the invariant set for tracking Ω_t^a is given:
$$\mathcal{Y}_s \triangleq \{y_s = N_\theta\theta : (x_s, u_s) = M_\theta\theta, \text{and } (x_s, \theta) \in \Omega_t^a\}$$
This set is potentially the set of all admissible outputs for system (3.1) subject to (3.2), (Limon et al., 2008a).

Taking into account the proposed conditions on the controller parameters, in the following theorem asymptotic stability and constraints satisfaction of the controlled system are proved .

Theorem 3.4 (Stability) *Consider that assumptions 3.1, 3.2 and 3.3 hold and consider a given target operation zone Γ_t. Then for any feasible initial state $x_0 \in \mathcal{X}_N$, the system controlled by the proposed MPC controller $\kappa_N^Z(x, \Gamma_t)$ is stable, fulfils the constraints throughout the time evolution and, besides*

(i) If $\Gamma_t \cap \mathcal{Y}_s \neq \emptyset$ then the closed-loop system asymptotically converges to a steady output $y(\infty) \in \Gamma_t$.

(ii) If $\Gamma_t \cap \mathcal{Y}_s = \emptyset$, the closed-loop system asymptotically converges to a steady output $y(\infty) = y_s^*$, such that

$$y_s^* \triangleq \arg \min_{y_s \in \mathcal{Y}_s} V_O(y_s, \Gamma_t)$$

Proof: The proof of this theorem follows the same argument as the one of Theorem 1 in Chapter 2, since the offset cost function $V_O(y_s, \Gamma_t)$ is convex, as stated in assumption 3.2. ■

3.4 Properties of the proposed controller

3.4.1 Steady state optimization

In practice it is not unusual that the zones chosen as target sets are not fully consistent with the model and, thus, fully or partly unreachable. This may happen when no point in the zone is an admissible operating point for the system.
From the latter theorem it can be clearly seen that in this case, the proposed controller steers the system to the optimal operating point according to the offset cost function $V_O(y_s, \Gamma_t)$. Then it can be considered that the proposed controller has a steady state optimizer built in and $V_O(y_s, \Gamma_t)$ defines the function to optimize.

3.4.2 Feasibility for any reachable target zone

The controller is able to guarantee feasibility for any Γ_t and for any prediction horizon N. Then, it can be derived that the proposed controller is able to track any admissible target zone (i.e. $\Gamma_t \cap \mathcal{Y}_s \neq \emptyset$) even for $N = 1$, if the system starts from an admissible equilibrium point. Nevertheless, a prediction horizon $N > 1$ is always a better choice, because, if on one hand a small prediction horizon reduces the computational effort, on the other hand the performances of the controller improve with N increasing.

3.4.3 Changing target zones

Taking into account theorem 3.4, stability is proved for any offset cost function satisfying assumption 3.2. Since the set of constraints of $P_N^Z(x,\Gamma_t)$ does not depend on Γ_t, its feasibility region does not depend on the target zone Γ_t. Therefore, if Γ_t varies with the time, the results of the theorem still hold. This property will be shown in the example.

3.4.4 Input target

The zone control problem can be formulated considering input targets u_t that must satisfy some constraint (i.e. $u_{min} \leq u_t \leq u_{max}$) to allow the outputs to be inside of a certain zone (Wang, 2002). These input targets are basically specific values for the inputs that are desirable to achieve for economic reasons. The proposed controller can be formulated considering input targets by defining an offset cost function $V_O(u_s, \Gamma_{u_t})$ subdifferential and convex w.r.t. u_s, where Γ_{u_t} is a convex polyhedron.
Moreover, all the results and properties of the proposed controller remain valid because this case is equivalent to considering $C = 0$ and $D = I$.

3.4.5 Enlargement of the domain of attraction

The domain of attraction of the MPC is the set of states that can be admissible steered to Ω_t in N steps. The fact that this set is an invariant set for any equilibrium points makes this set (potentially) larger than the one calculated for regulation to a fixed equilibrium point. Consequently, the domain of attraction of the proposed controller is (potentially) larger than the domain of the standard MPC. This property is particularly interesting for small values of the control horizon.

3.4.6 Terminal constraint

The optimization problem $P_N^Z(x,\Gamma_t)$ can also be formulated by posing the terminal constraint as a terminal equality constraint, by considering $P = 0$ and Ω_t^a such that:

$$\Omega_t^a \triangleq \{(x,\theta) : M_\theta\theta \in \mathcal{Z}, \quad x = M_x\theta\}$$

3.4.7 Convexity of the optimization problem

Since all the ingredients (functions and sets) of the optimization problem $P_N^Z(x, \Gamma_t)$ are convex, then it derives that $P_N^Z(x, \Gamma_t)$ is a convex mathematical programming problem that can be efficiently solved in polynomial time by specialized algorithms (Boyd and Vandenberghe, 2006).

3.5 Formulations of the MPC for tracking target sets leading to QP problems

It is clear from the previous sections that one of the results of the controller presented in this chapter, is the concept of distance from a set. The optimization problem $P_N^Z(x, \Gamma_t)$ is a convex mathematical programming problem that can be efficiently solved by specialized algorithms (Boyd and Vandenberghe, 2006). From a practical point of view, it is desirable that, even considering a distance from a set as an *offset cost* function, the optimization problem $P_N^Z(x, \Gamma_t)$ still remains a Quadratic Programming problem. To this aim, in this section, three different implementations of the MPC for tracking with target sets are presented, which ensures that the optimization problem can be formulated as a QP problem.

Consider the target set Γ_t and define as y_t a specific point that belongs to the zone region, typically the center of the zone. As it has been stated in theorem 3.4, in the problem of tracking a target set, three situations can be addressed.

a) There not exists an admissible steady output in the zone, i.e. $\Gamma_t \cap \mathcal{Y}_s = \emptyset$.

b) There exists an admissible steady state in the zone, but the desired output is not admissible, i.e. $\Gamma_t \cap \mathcal{Y}_s \neq \emptyset$ and $y_t \notin \mathcal{Y}_s$.

c) There exists an admissible steady state in the zone and the desired output is admissible, i.e. $\Gamma_t \cap \mathcal{Y}_s \neq \emptyset$ and $y_t \in \mathcal{Y}_s$.

These three situations are shown in figure 3.1 where the double integrator system presented in (Limon et al., 2008a) has been considered. This system is given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0 & 0.5 \\ 1.0 & 0.5 \end{bmatrix}, \text{ and } \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

which is constrained to $\|x\|_\infty \leq 5$ and $\|u\|_\infty \leq 0.3$. In the picture, the domain of attraction $\mathcal{X}_N$ for $N = 3$, the invariant set for tracking Ω_t, and the region of admissible steady state $\mathcal{X}_s$ are depicted respectively in black solid line, black dashed line and red line. Notice that $\mathcal{X}_s \equiv \mathcal{Y}_s$, since $C = \mathbf{I}_2$. The three target set situations previous mentioned are represented by the three boxes labeled as a), b) ad c). The center of each box, depicted as a circle, is the desirable target point into the zone region, y_t. In particular the zone region and the desirable target point for each case are:

a) $\Gamma_t = \{1 \leq y_1 \leq 2.6, -1.9 \leq y_2 \leq -1.1\}$ and $y_t = (1.8, -1.5)$.

b) $\Gamma_t = \{-1.65 \leq y_1 \leq -0.05, -0.9 \leq y_2 \leq -0.1\}$ and $y_t = (-0.85, -0.5)$.

c) $\Gamma_t = \{-4.3 \leq y_1 \leq -2.7, -0.45 \leq y_2 \leq -0.35\}$ and $y_t = (-3.5, -0.05)$.

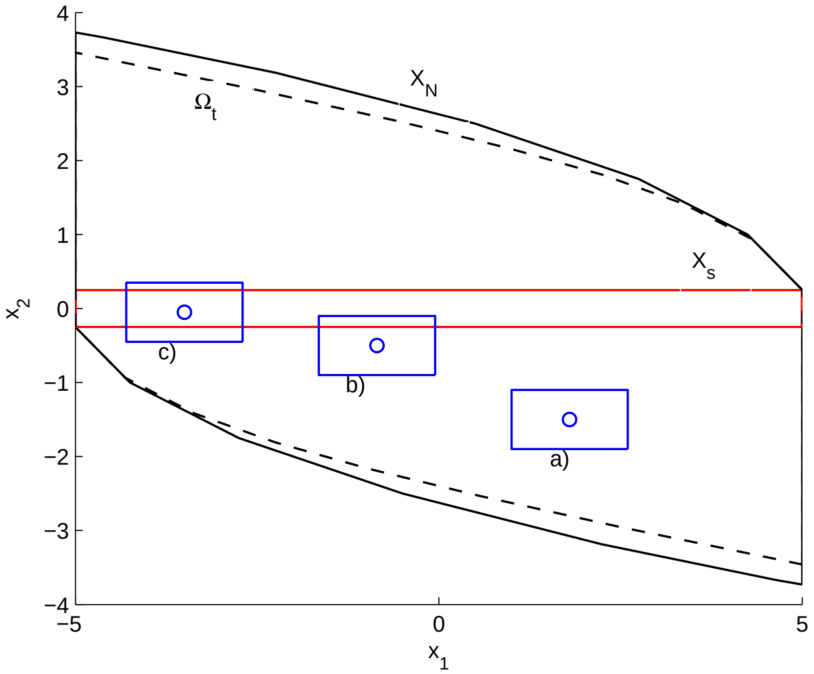


Figure 3.1: Target sets for the double integrator system.

The controller presented in this chapter will steer the system to that point which minimizes the offset cost function. This point can be a point belonging to $\mathcal{Y}_s$ (case a)) or a point belonging to the intersection of $\mathcal{Y}_s$ with Γ_t (cases b) and c)). The controller implementation presented in section 3.5.3, in the case c), will steer the system exactly to the desired setpoint y_t.

3.5.1 Distance from a set: ∞-norm

Consider that Γ_t is a set-interval zone defined as

$$\Gamma_t \triangleq \{y : y_{min} \leq y \leq y_{max}\}$$

where the inequality is component-wise. Define as y_t the desirable target point into the zone region, typically the center of the zone.

In this implementation, the offset cost function is chosen as the distance from y_s to the target region Γ_t, measured by a ∞-norm. Hence, the offset cost function $V_O(y_s, \Gamma_t)$ is given by:

$$V_O(y_s, \Gamma_t) \triangleq \min_{y \in \Gamma_t} \|y_s - y\|_\infty$$

Consider the following lemma:

Lemma 3.5 *(Rockafellar, 1970) The set* $\Xi \triangleq \{y_s : \min\limits_{y \in \Gamma_t} \|y_s - y\|_\infty \leq \lambda\}$ *is given by*

$$\begin{aligned} y + \lambda \mathbf{1} &\leq y_{max} \\ -y - \lambda \mathbf{1} &\leq -y_{min} \\ \lambda &\geq 0 \end{aligned}$$

where $\mathbf{1} \in \mathbb{R}^p$ *is a vector of all unitary elements.*

Thanks to this lemma, and considering the offset cost function in its epigraph form, the optimization problem $P_N^Z(x, \Gamma_t)$ can be posed as a standard quadratic programming problem, by adding a new decision variable λ, such that

$$V_O(y_s, \Gamma_t) \leq \lambda$$

.

Thanks to the previous statements, the cost function can be written in the form:

$$V_N^Z(x,\Gamma_t;\mathbf{u},\theta,\lambda)\triangleq\sum_{i=0}^{N-1}\|x(i)-x_s\|_Q^2+\|u(i)-u_s\|_R^2+\|x(N)-x_s\|_P^2+\lambda$$

where λ is a new optimization variable, and the optimization problem $P_N^Z(x,\Gamma_t)$ is posed as:

$$\begin{aligned} V_N^{Z,0}(x,\Gamma_t) = & \min_{\mathbf{u},\theta,\lambda} V_N^Z(x,\Gamma_t;\mathbf{u},\theta,\lambda) \\ & s.t. \ (3.4b),(3.4c),(3.4d),(3.4e),(3.4f),(3.4g) \\ & \quad y_s+\lambda\mathbf{1}\leq y_{max} \\ & \quad -y_s-\lambda\mathbf{1}\leq -y_{min} \\ & \quad \lambda\geq 0 \end{aligned}$$

which is a formulation of $P_N^Z(x,\Gamma_t)$ as a QP problem.

In figure 3.2 the trajectories for the double integrator system, from the initial state $x_0=(-3,2)$, for the three situations above mentioned, using a ∞-norm distances are plotted.

See how the controller steers the system to the point that minimize the ∞-norm distance. In particular, see that in cases b) and c) the system converges to a point inside the zone regions. The role of the ∞-norm is important in cases such a). In this case, in fact, the system converges to one of those points that minimize the ∞-norm distance from the target region.

3.5.2 Distance from a set: 1-norm

Consider that Γ_t is a set-interval zone defined as

$$\Gamma_t\triangleq\{y: y_{min}\leq y\leq y_{max}\}$$

Define as y_t the desirable target point into the zone region, typically the center of the zone.

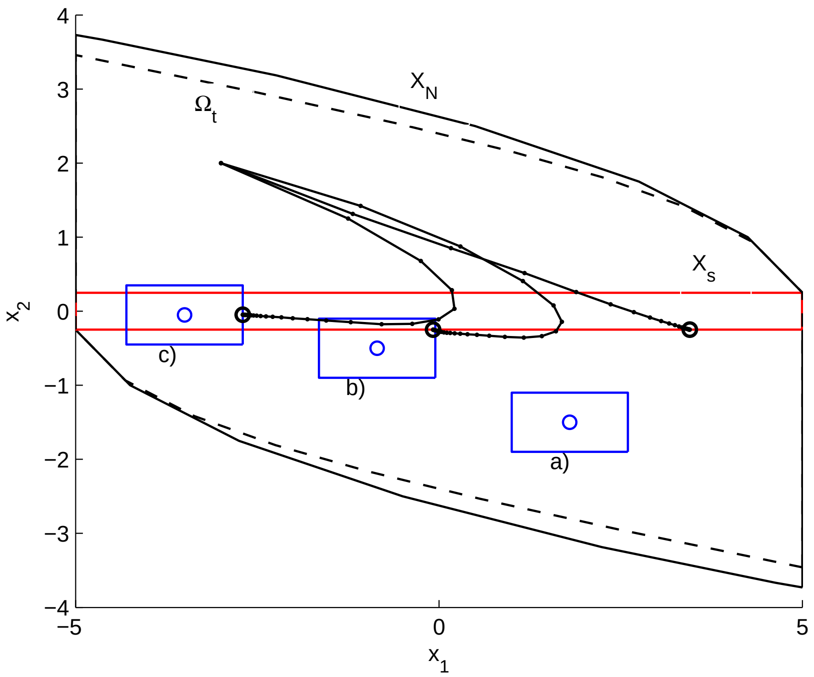


Figure 3.2: The double integrator system: ∞-norm distance.

In this implementation, the offset cost function is chosen as the distance from y_s to the target region Γ_t, measured using a 1-norm:

$$V_O(y_s, \Gamma_t) \triangleq \min_{y \in \Gamma_t} \|y_s - y\|_1$$

As in the previous case, the optimization problem $P_N^Z(x, \Gamma_t)$ can be posed as a standard quadratic programming problem, by considering the offset cost function in its epigraph form $V_O(y_s, \Gamma_t) \leq \lambda$ and by resorting the following lemma.

Lemma 3.6 *(Rockafellar, 1970) The set* $\Xi \triangleq \{y_s : \min_{y \in \Gamma_t} \|y_s - y\|_1 \leq \lambda\}$ *is given by*

$$\begin{aligned} \mathbf{1}'y + \lambda &\leq \mathbf{1}'y_{max} \\ -\mathbf{1}'y - \lambda &\leq -\mathbf{1}'y_{min} \\ \lambda &\geq 0 \end{aligned}$$

The cost function to minimize is given by (3.5) and the optimization problem

$P_N^Z(x, \Gamma_t)$ is given by:

$$\begin{aligned} V_N^{Z,0}(x,\Gamma_t) = \min_{\mathbf{u},\theta,\lambda} & \; V_N^Z(x,\Gamma_t;\mathbf{u},\theta,\lambda) \\ s.t. & \; (3.4b),(3.4c),(3.4d),(3.4e),(3.4f),(3.4g) \\ & \; \mathbf{1}'y_s + \lambda \leq \mathbf{1}'y_{max} \\ & \; -\mathbf{1}'y_s - \lambda \leq -\mathbf{1}'y_{min} \\ & \; \lambda \geq 0 \end{aligned}$$

where λ is a new optimization variable, and which is a formulation of $P_N^Z(x, \Gamma_t)$ as a QP problem.

In figure 3.3 the trajectories for the double integrator system, from the initial state $x_0 = (-3, 2)$, for the three situations above mentioned, using a 1-norm distances are plotted.

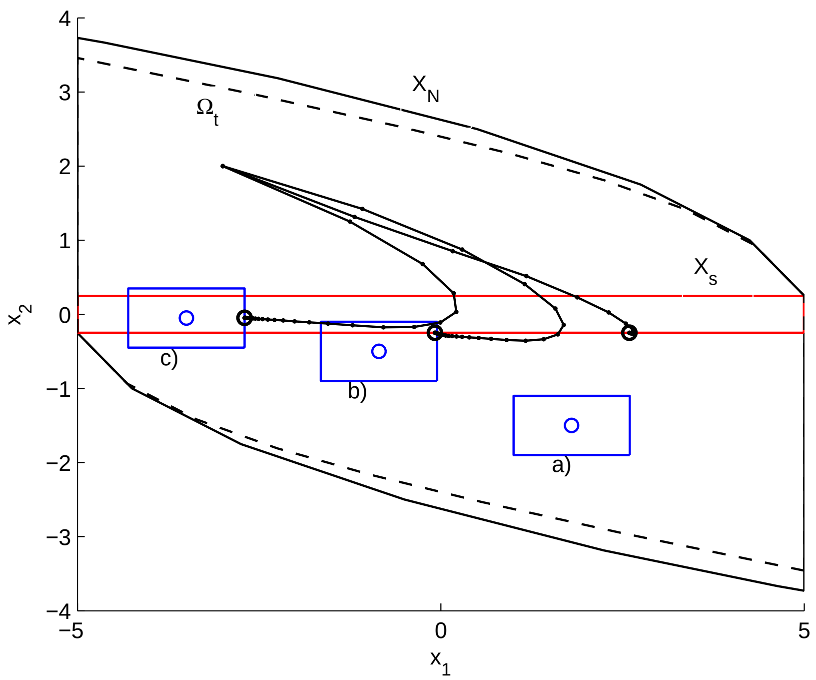


Figure 3.3: The double integrator system: 1-norm distance.

See how the controller steers the system to the point that minimize the 1-norm distance. In particular, see that in cases $b)$ and $c)$ the system converges to a point inside the zone regions. The role of the norm is important in this case a). In this case, the system converges to one of those points that minimize the 1-norm distance (see also figure 3.2).

3.5.3 Scaling factor

In this implementation, the target region is defined as

$$\Gamma_t \triangleq y_t \oplus \Xi_t$$

where y_t is a desired target point and Ξ_t is a polyhedron that defines the zone. The offset cost function $V_O(y_s, \Gamma_t)$ is chosen as a kind of distance from y_s to the target region Γ_t, given by

$$\begin{aligned} V_O(y_s, \Gamma_t) \quad &= \quad \min_{\lambda, y} \lambda \\ s.t. \quad & \lambda \geq 0 \\ & y - y_t \in \lambda \Xi_t \end{aligned}$$

This measure is such that, if $y \notin \Gamma_t$ then $\lambda > 1$, and if $y \in \Gamma_t$ then $\lambda \in [0, 1]$. In particular, if $y = y_t$, hence $\lambda = 0$. Therefore, λ has the double role of measuring the distance to a set and to a point.

In order to formulate the optimization problem as a QP, the cost function is chosen as in (3.5) and is minimized considering the following constraint:

$$y_s - y_t \in \lambda \Xi_t$$

with $\lambda \geq 0$. This means that y_s should remain in a zone that is an homothetic transformation of Γ_t centered in y_t.

Then, the optimization problem $P_N^Z(x, \Gamma_t)$ is given by:

$$\begin{aligned} V_N^{Z,0}(x, \Gamma_t) = & \min_{\mathbf{u}, \theta, \lambda} V_N^Z(x, \Gamma_t; \mathbf{u}, \theta, \lambda) \\ s.t. \; & (3.4b), (3.4c), (3.4d), (3.4e), (3.4f), (3.4g) \\ & y_s - y_t \in \lambda \Xi_t \\ & \lambda \geq 0 \end{aligned}$$

where λ is an optimization variable, and which is a formulation of $P_N^Z(x, \Gamma_t)$ as a QP problem. Notice that problem $P_N^Z(x, \Gamma_t)$ is a QP problem, for any Ξ_t that is a convex polyhedron.

In figure 3.4 the trajectories for the double integrator system, from the initial state $x_0 = (-3, 2)$, for the three situations above mentioned, using the homothetic transformation method are plotted.

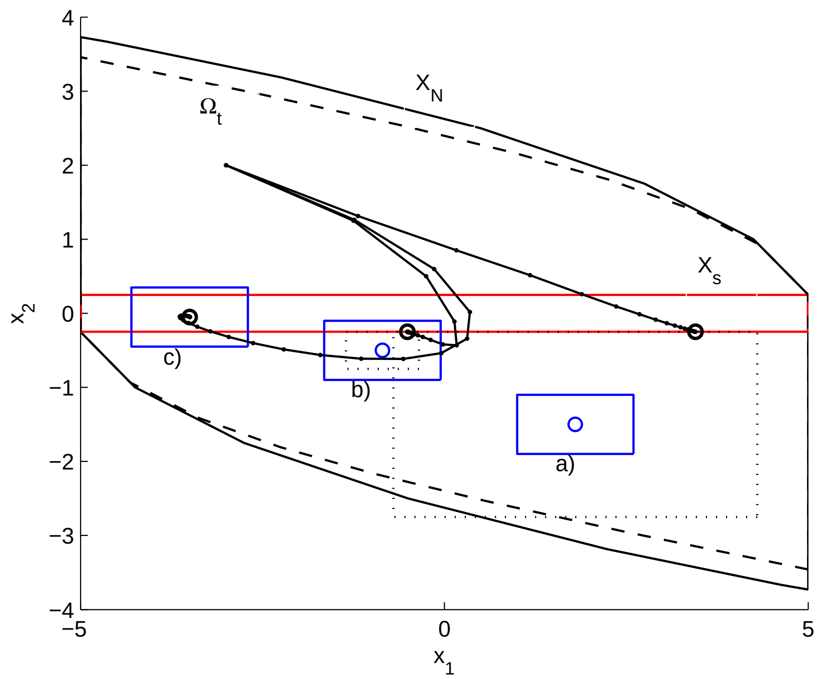


Figure 3.4: The double integrator system: homothetic transformation.

The zone regions are depicted in solid line while their homothetic transformation are depicted in dotted line. Notice that, when $y_t \in \Gamma_t \cap \mathcal{Y}_s$, the homothetic transformation of Γ_t is the target point y_t. See how the controller steers the system to the point that minimize the offset cost function w.r.t. the homothetic transformation.

3.6 Illustrative example

In this section, an example to test the performance of the proposed controller, is presented. The system adopted is the 4 tanks process presented in the Appendix A. The objective of the controller is to maintain the system within some specified zones.

The objective of the simulation is to show how the proposed controller manages a target set given by a combination of both, output setpoints and output zones. To this aim, five changes of these target sets have been considered (see table 3.1, where (h_1^0, h_2^0) is the point around which the system has been linearized defined in the Appendix A), that are in fact changes of the zones into which the outputs should be steered. In particular, in the third change of reference, we considered the case in which both target set and desirable setpoint are not admissible ($\Gamma_t \cap \mathcal{Y}_s = \emptyset$ and $y_t \notin \mathcal{Y}_s$), while the case in which both target set and desirable setpoint are admissible is considered in the other cases ($\Gamma_t \cap \mathcal{Y}_s \neq \emptyset$ and $y_t \in \mathcal{Y}_s$).

Table 3.1: Target zones used in the example

Γ_t	y_{min}	y_{max}
$\Gamma_{t,1}$	$(0.20, 0.20)$	$(0.40, 0.40)$
$\Gamma_{t,2}$	$(1.15, 1.15)$	$(1.35, 1.35)$
$\Gamma_{t,3}$	$(0.30, 0.75)$	$(0.40, 0.85)$
$\Gamma_{t,4}$	$(0.95, 0.75)$	$(1.05, 0.85)$
$\Gamma_{t,5}$	$(h_1^0, h_2^0) - (0.05, 0.05)$	$(h_1^0, h_2^0) + (0.05, 0.05)$

It is convenient to remark that, inside the zones, there are no preferences between one point and another. Moreover, an other objective of the example is to show the 3 different implementations of the controller, presented in section 3.5.

3.6.1 Distance from a set: ∞-norm

In this section, the results of the simulations for the ∞-norm distance from a set implementation of the offset cost function are presented. Figure 3.5 shows the state-space evolution of the system.

The projection of the domain of attraction onto y, $\mathcal{Y}_N$ for $N = 3$, the projection of the invariant set for tracking $\Omega_y = Proj_y(\Omega_t)$, and the region of admissible steady outputs $\mathcal{Y}_s$ are depicted in solid and dashes dotted line. The zone regions are represented as boxes, and the desirable target points y_t, are represented as circles and considered as the center of the target zones.

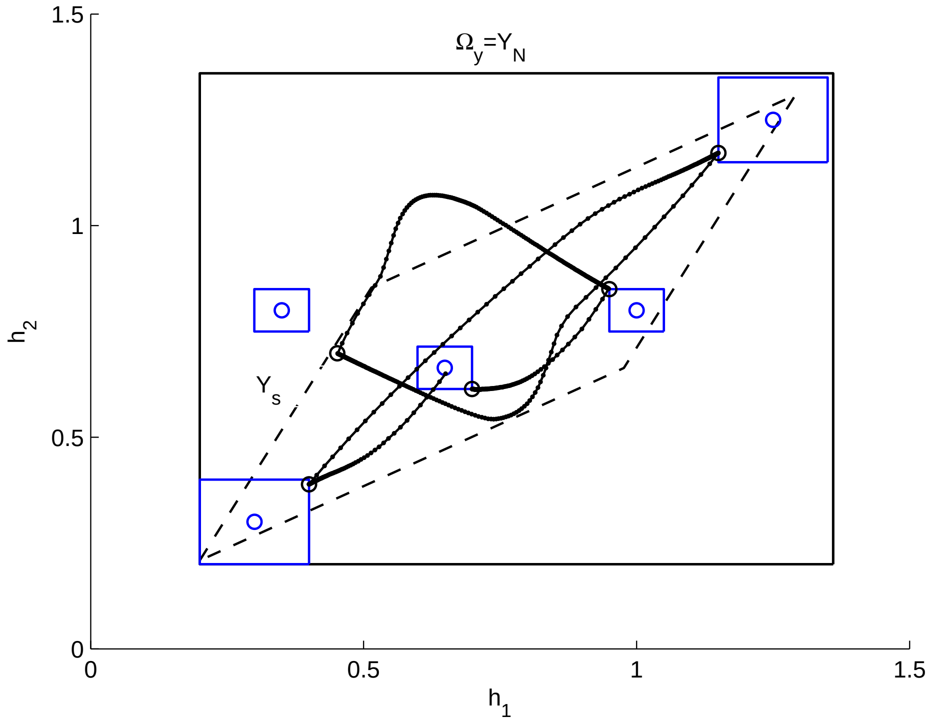


Figure 3.5: State-space evolution for the ∞-norm formulation.

In Figure 3.6 and 3.7 the time evolution of h_1, h_2, h_3, h_4, q_a and q_b is depicted.

The evolutions of the outputs and the artificial references are drawn respectively in solid and dashed line. The zones are drawn in dotted lines. See how the controller steers (whenever possible) the system into the output zone, even if the initial condition stays out of the zone. Furthermore, if the output zone is not admissible, that is $\Gamma_t \cap \mathcal{Y}_s = \emptyset$, the controller steers the system to the admissible point that minimizes the offset cost function. This can be seen in the third output zone change (from sample 500 to 750), in which the outputs h_1 and h_2 are steered to stationary value out of the corresponding zones. In the other cases, it can be seen that the controller steers the outputs into the zones. This happens because $\Gamma_t \cap \mathcal{Y}_s \neq \emptyset$ and $y_t \in \mathcal{Y}_s$. Furthermore, and despite it was not simulated, the proposed algorithm also allows the possibility to include input target, i.e., specific values for the inputs that are desirable to achieve for economic reasons.

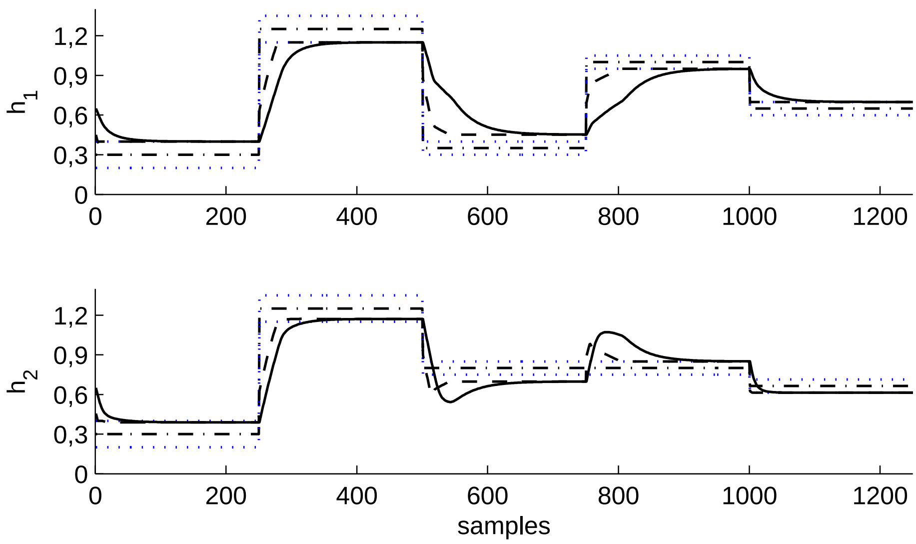


Figure 3.6: ∞-norm formulation: evolution of h_1 and h_2.

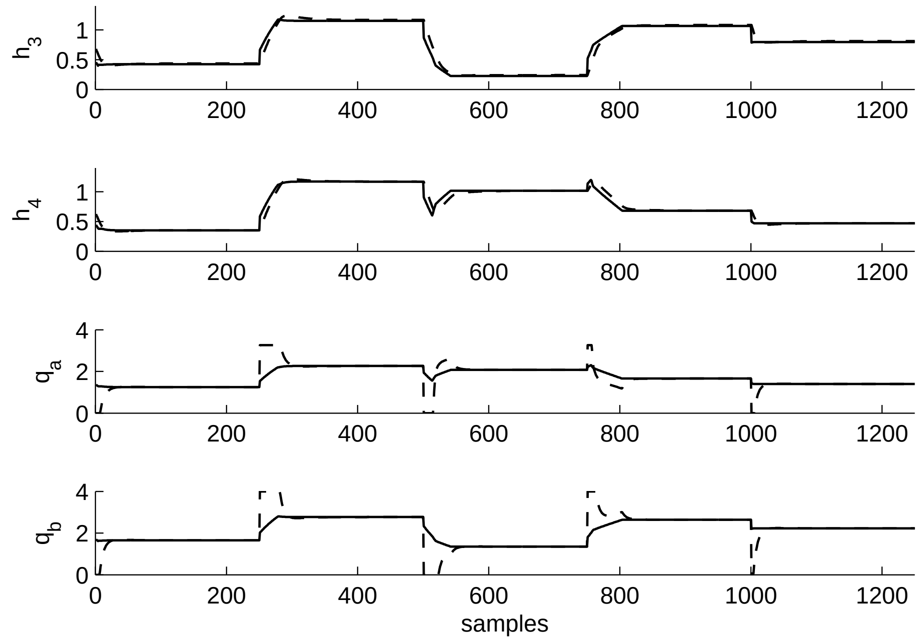


Figure 3.7: ∞-norm formulation: evolution of h_3, h_4 q_a and q_b.

3.6.2 Distance from a set: 1-norm

In this example, the controller is set-up for considering a 1-norm as offset cost function. The results of the simulations are presented in figure 3.8, which shows the state-space evolution of the system, and in figures 3.9 and 3.10, which show the time evolution of the outputs and of the inputs.

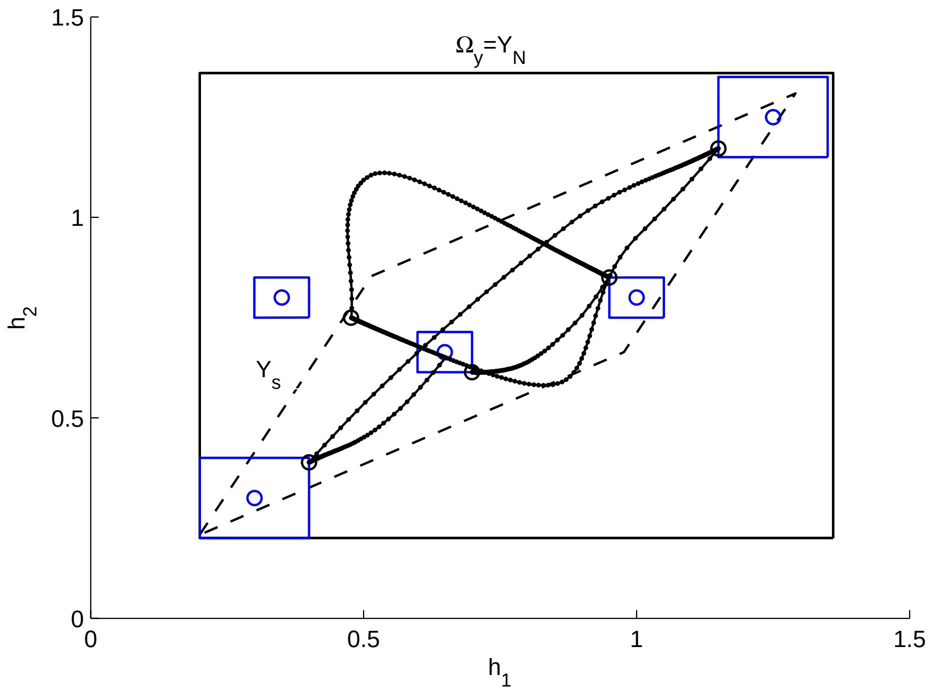


Figure 3.8: State-space evolution for the 1-norm formulation.

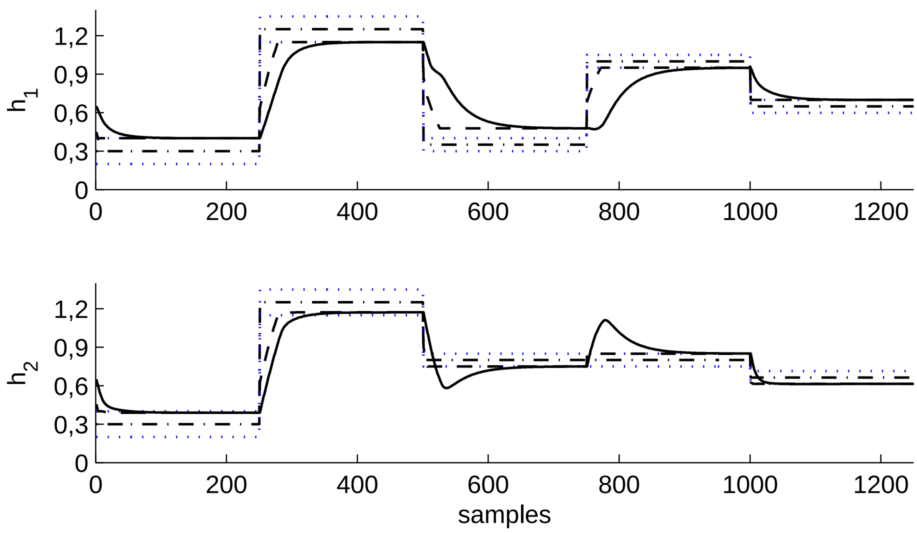


Figure 3.9: 1-norm formulation: evolution of h_1 and h_2.

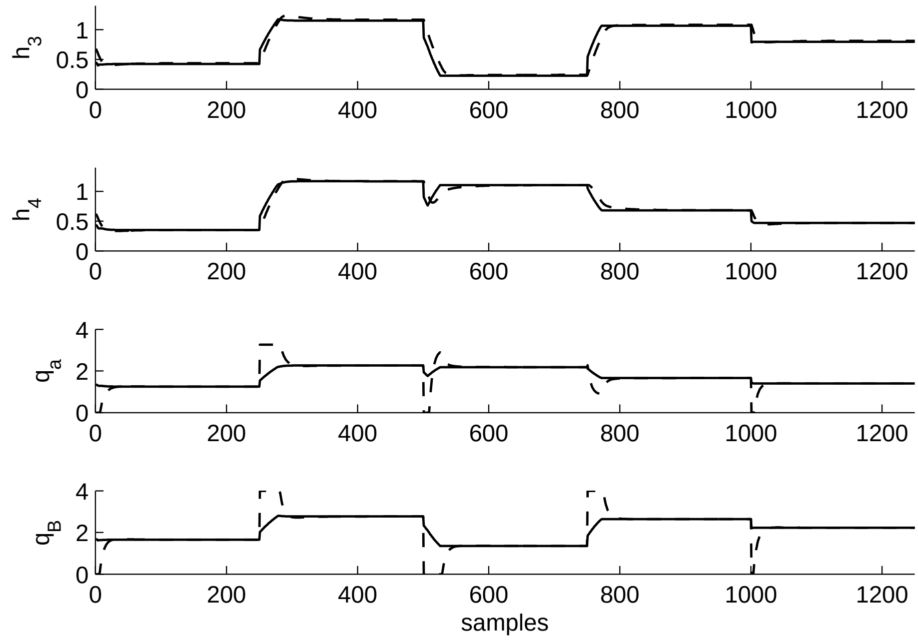


Figure 3.10: 1-norm formulation: evolution of h_3, h_4 q_a and q_b.

In Figure 3.8 the projection of the domain of attraction onto y, $\mathcal{Y}_N$ for $N = 3$, the projection of the invariant set for tracking $\Omega_y = Proj_y(\Omega_t)$, and the region of admissible steady outputs $\mathcal{Y}_s$ are depicted in solid and dashes dotted line. The zone regions are represented as boxes, and the desirable target points y_t, are represented as circles and considered as the center of the target zones. In figures 3.9 and 3.10, the time evolution of h_1, h_2, h_3, h_4, q_a and q_b is depicted. The evolutions of the outputs and the artificial references are drawn respectively in solid and dashed line. The zones are drawn in dotted lines. As in the previous case, the controller steers (whenever possible) the system into the output zone, even if the initial condition lies out of the zone. In the third output zone change (from sample 500 to 750), it can be seen how the outputs are steered to a stationary value out of the corresponding zones, which is the one that minimizes the offset cost function. This happens because the target zone is not admissible ($\Gamma_t \cap \mathcal{Y}_s = \emptyset$). In the other cases, the controller steers h_1 and h_2 into the zone. This happens because $\Gamma_t \cap \mathcal{Y}_s \neq \emptyset$ and $y_t \in \mathcal{Y}_s$. This formulation, as the previous one, can also cope with input targets.

3.6.3 Scaling factor

The last controller implementation proposed in section 3.5, the scaling factor, is presented in this section. Figures 3.11, 3.12 and 3.13 show the result of this case's simulation, which are respectively the state-space evolution of the system and the time evolution of outputs and inputs.

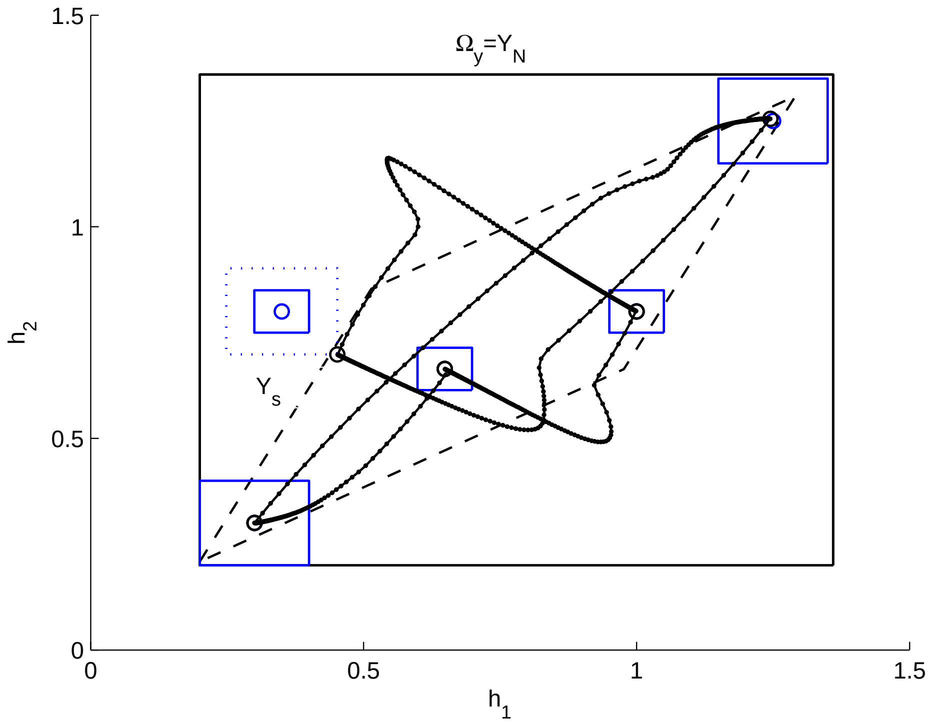


Figure 3.11: State-space evolution for the scaling factor formulation.

As in the two previous cases, in figure 3.11 the projection of the domain of attraction onto y, $\mathcal{Y}_N$ for $N = 3$, the projection of the invariant set for tracking $\Omega_y = Proj_y(\Omega_t)$, and the region of admissible steady outputs $\mathcal{Y}_s$ are depicted in solid and dashes dotted line. The polyhedra Ξ_t that define the zone regions are represented as boxes, and the desirable target points y_t, are represented as circles and considered as the center of the target zones. In Figure 3.12 and 3.13, the time evolution of h_1, h_2, h_3, h_4, q_a and q_b is depicted. The evolutions of the outputs and the artificial references are drawn respectively in solid and dashed-dotted line. The zones are drawn in thick-solid lines. The main difference between this implementation and the previous is that when $\Gamma_t \cap \mathcal{Y}_s \neq \emptyset$ and $y_t \in \mathcal{Y}_s$ the controller steers the system to exactly y_t, while if

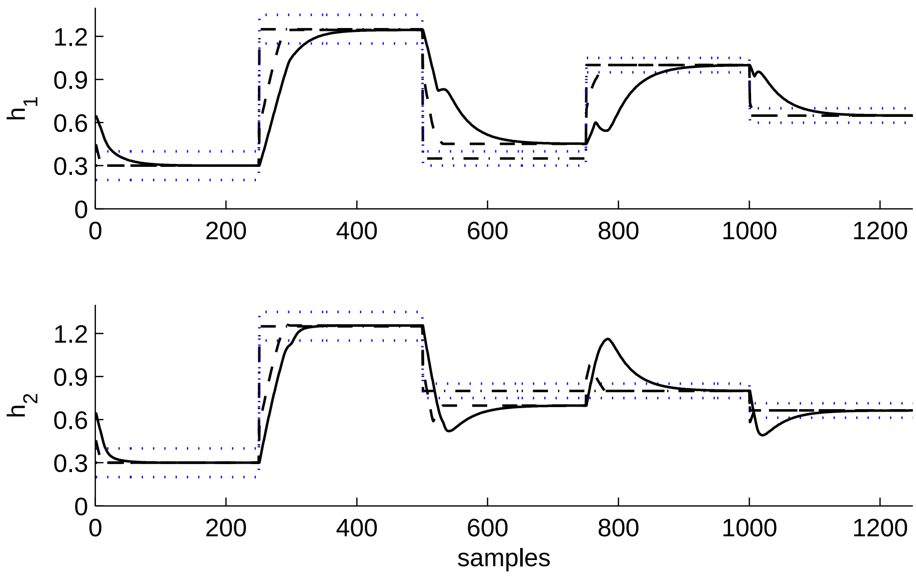


Figure 3.12: Scaling factor formulation: evolution of h_1 and h_2.

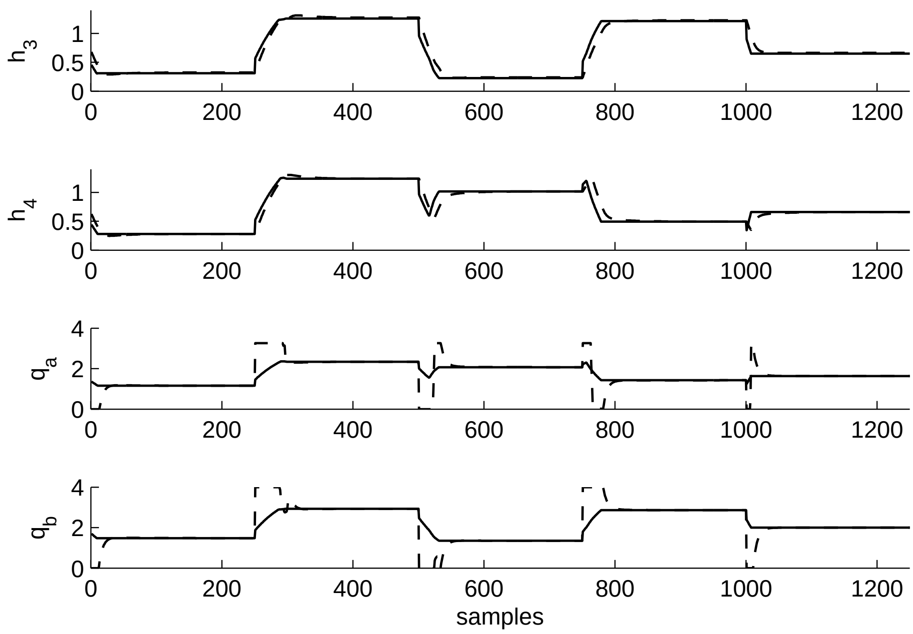


Figure 3.13: Scaling factor formulation: evolution of h_3, h_4 q_a and q_b.

$\Gamma_t \cap \mathcal{Y}_s = \emptyset$ (third reference), the controller steers the system to the admissible point that minimizes the offset cost function. This last simulation shows that the controller account for the frequent practical case in which a combination of output set-point and zones is given. This last implementation allows the possibility to include input target, as well.

3.7 Conclusions

The zone control strategy is implemented in applications where the exact values of the controlled outputs are not important, as long as they remain inside a range with specified limits. In this chapter, an extension of the MPC for tracking for zone control has been presented, in which the controller considers a set, instead of a point, as target. The concept of deviation between two points used in the offset cost function has been generalized to the concept of distance from a point to a set. A characterization of the offset cost function has been given as the minimal distance between the output and some point inside the target set. Three different formulations of the offset cost function have been proposed, to obtain a QP problem.

CHAPTER 4

Robust MPC for tracking target sets based on nominal predictions

4.1 Introduction

This chapter deals with the problem of robust tracking of target sets for constrained uncertain linear systems. Usually, in case of robust MPC problems, the control objective is to ensure stability despite the uncertainties and robust constraint satisfaction while a certain performance index is optimized. At the same time, dealing with a tracking problem requires feasibility to be guaranteed under any setpoint change.

Several solutions have been proposed in literature to solve the robust tracking problem. In (Rossiter et al., 1996; Chisci and Zappa, 2003) an auxiliary controller that is able to recover the feasibility in finite time is used leading to a switching strategy. The controllers proposed in (Pannocchia and Kerrigan, 2005; Pannocchia, 2004) are based on the robust MPC proposed in (Chisci et al., 2001) but consider the change of the setpoint as a disturbance to be rejected; thus, this technique is able to steer the system to the desired setpoint, but only when the variations of the setpoint are small enough; so this solution results to be conservative.

A different approach has been proposed in the context of the reference governors (Gilbert et al., 1999; Bemporad et al., 1997). This control technique assumes that the system is robustly stabilized by a local controller, and a nonlinear filtering of the reference is designed to ensure the robust satisfaction of the constraints. These controllers ensure robust tracking without considering the performance of the obtained controller.

A recent approach to the design of robust MPC for constrained linear systems is the so-called tube-based robust MPC (Langson et al., 2004; Mayne et al., 2005). In this case, using a suitable compensation of the effect of the uncertainty based on the nominal trajectory, a tube containing the uncertain trajectories is calculated. This is centered in the nominal trajectory and its section is the minimum robust invariant set. This property allows to formulate the robust MPC control problem as the solution of a optimization control problem based on nominal predictions.

This robust MPC formulation has been extended to the case of MPC for tracking set-points in (Limon et al., 2010a). The controller has demonstrated to be a nice solution to the robust control problem, but the main drawback found is the calculation of the minimum robust invariant set, since its complexity grows exponentially with the dimension of the system.

In this chapter, a robust MPC based on nominal predictions is presented. The controller presented in (Ferramosca et al., 2010a) is extended to cope with the problem of robust tracking of target sets in presence of additive disturbance. The proposed controller uses the results presented in (Chisci et al., 2001), in which an MPC based on nominal predictions and restricted constraints is presented, which ensures stability, robust satisfaction of the constraints en recursive feasibility. The plant is assumed to be modeled as a linear system with additive uncertainties confined to a bounded known polyhedral set. Remarkably, this robust control does not require the calculation of the minimum robust invariant set and the obtained properties result to be similar to the ones of the tube-based robust controller.

The derived controller, under mild assumptions, is feasible under any change of the controlled variables target and steers the uncertain system to (a neighborhood of) the target if this is admissible. If the target is not admissible, the system is steered to the closest admissible operating point.

4.2 Problem statement

Consider a plant described by the following uncertain discrete-time LTI system

$$\begin{aligned} x^+ &= Ax + Bu + w \\ y &= Cx + Du \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state of the system at the current time instant, x^+ denotes the successor state, that is, the state of the system at next sampling time, $u \in \mathbb{R}^m$ is the manipulated control input, $y \in \mathbb{R}^p$ is the controlled variables and $w \in \mathbb{R}^n$ is an unknown but bounded state disturbance. In what follows, $x(k)$, $u(k)$, $y(k)$ and $w(k)$ denote the state, the manipulable variable, controlled variable and the disturbance respectively, at sampling time k.

The plant is subject to hard constraints on state and control:

$$(x(k), u(k)) \in \mathcal{Z} \tag{4.2}$$

where $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$ is a compact convex polyhedron containing the origin in its interior.

Define also the plant nominal model, given by (4.1) neglecting the disturbance input w:

$$\begin{aligned} \bar{x}^+ &= A\bar{x} + B\bar{u} \\ \bar{y} &= C\bar{x} + D\bar{u} \end{aligned} \tag{4.3}$$

The plant model is assumed to fulfil the following assumption:

Assumption 4.1

- *The pair (A, B) is controllable.*
- *The uncertainty vector w is bounded and lies in a compact convex polyhedron containing the origin in its interior*

  $$\mathcal{W} = \{w \in \mathbb{R}^n : A_w w \leq b_w\} \tag{4.4}$$

  that is, $w(k) = \Big(x(k+1) - Ax(k) - Bu(k)\Big) \in W$ for all $(x(k), u(k)) \in \mathcal{Z}$.
- *The state of the system is measured, and hence $x(k)$ is known at each sample time.*

It is remarkable that no assumption is considered on the number of inputs m and outputs p, allowing thin plants $(p > m)$, square plants $(p = m)$ and flat plants $(p < m)$. Moreover, it is not assumed that (A, B, C, D) is a minimal realization of the state-space model. This allows us to use state-space models derived from input-output models, that is, using as state a collection of past inputs and outputs of the plant (Camacho and Bordons, 2004). The necessity of an observer is also avoided while the global uncertainty and the noise can be posed as additive uncertainties in the state-space model (4.1).

The aim of this chapter is to find a control law $u(k) = \kappa_N(x(k), \Gamma_t)$ such that the system is steered into a (possibly time varying) region Γ_t, which defines the range into which the controlled outputs should remain fulfilling the plant constraints $(x(k), u(k)) \in \mathcal{Z}$, despite the uncertainties.

4.3 Robust MPC with restricted constraints

In this section, the robust MPC for tracking with restricted constraints (Chisci et al., 2001) is briefly introduced. In this robust MPC formulation the keystone is to use predictions based on the nominal system, that is neglecting the disturbance input w, and to restrict the constraints set $\mathcal{X}$ and $\mathcal{U}$, by subtracting a robust positive invariant set, at any step of the prediction horizon.

This controller is based on a pre-stabilization of the plant using a state feedback control gain K, such that $A_K = A + BK$ has all its eigenvalues in the unit circle. The controlled system is then given by

$$\begin{aligned} x(k+1) &= A_K x(k) + Bc(k) + w(k) \\ u(k) &= Kx(k) + c(k) \end{aligned}$$

The notion of robust positively invariant (RPI) set (Kolmanovsky and Gilbert, 1998; Rakovic et al., 2005) plays an important role in the design of robust controllers for constrained systems. This is defined as follows:

Definition 4.2 *A set Ω is called a robust positively invariant (RPI) set for the uncertain system $x(k+1) = A_K x(k) + w(k)$ with $w(k) \in \mathcal{W}$ if $A_K\Omega \oplus \mathcal{W} \subseteq \Omega$.*

It will be also useful to define the so-called reachable sets, that are outer bounds of the forced response of the system due to the uncertainty.

Definition 4.3 *The reachable set in j steps, $\mathcal{R}_j$, is given by*

$$\mathcal{R}_j \triangleq \bigoplus_{i=0}^{j-1} A_K^i \mathcal{W}$$

This is the set of states of the nominal closed-loop systems which are reachable in j steps from the origin, under the disturbance input w, (Chisci et al., 2001). This set satisfies the following properties:

(i) It is given by the recursion $\mathcal{R}_j \oplus A_K^j \mathcal{W} = \mathcal{R}_{j+1}$with $\mathcal{R}_1 = \mathcal{W}$.

(ii) $A_K \mathcal{R}_j \oplus \mathcal{W} = \mathcal{R}_{j+1} = \mathcal{R}_j \oplus A_K^j \mathcal{W}$

(iii) The sequence of sets $\mathcal{R}_j$ has a limit $\mathcal{R}_\infty$ as $j \to \infty$, and $\mathcal{R}_\infty$ is a robust positive invariant set.

(iv) $\mathcal{R}_\infty$ is the minimal RPI set.

The proposed robust MPC consider a set of restricted constraints on the nominal predictions in the optimization problem. These sets are given by:

$$\begin{aligned} \bar{\mathcal{X}}_j &\triangleq \mathcal{X} \ominus \mathcal{R}_j \\ \bar{\mathcal{U}}_j &\triangleq \mathcal{U} \ominus K\mathcal{R}_j \end{aligned}$$

It is important to introduce the following assumption

Assumption 4.4 *The sets $\bar{\mathcal{X}}_j$ and $\bar{\mathcal{U}}_j$ exist if and only if $\mathcal{R}_\infty \subset \mathcal{X}_j$ for all $j \geq 0$.*

Moreover, the cost function to minimize is defined as follows:

$$V_N^c(x; \mathbf{c}) = \sum_{j=0}^{N-1} \|c(j)\|_\Psi^2 \tag{4.5}$$

where $\mathbf{c} = \{c(0), c(1), ..., c(N-1)\}$ and $\Psi = \Psi' > 0$ is given by $\Psi = R + B'PB$. In (Chisci et al., 2001) and (Pannocchia and Rawlings, 2003) it is proved that, in the case that K is the gain of the LQR, minimizing $V_N^r(x; \mathbf{c})$ is equivalent to minimizing the following cost function

$$\tilde{V}_N^c(x; \mathbf{c}) = \sum_{j=0}^{N-1} \|\bar{x}(j)\|_Q^2 + \|\bar{u}(j)\|_R^2 + \|\bar{x}(N)\|_P^2 \tag{4.6}$$

where $\bar{x}(j)$ is the nominal prediction of the model for $\bar{u}(j) = K\bar{x}(j) + c(j)$; P is the unique solution of the Riccati equation.

$$(A+BK)'P(A+BK)-P=-(Q+K'RK)$$

In fact, the equivalence between cost (4.5) and (4.6) holds since

$$\tilde{V}_N^c(x;\mathbf{c}) = V_N^c(x;\mathbf{c}) + \|\bar{x}(0)\|_P^2$$

Then, taking $K = K_{LQR}$, minimizing the cost (4.5) is equivalent to minimize the cost of the predicted nominal trajectory.

The control objective is to design a nonlinear state feedback $k_N^r(x)$, such that the system robustly fulfills the constraints. Then the following optimization control problem $P_N^r(c)$ is proposed:

$$\begin{aligned}
\min_{\mathbf{c}} \quad & V_N^r(x;\mathbf{c}) \\
s.t. \quad & \bar{x}(0) = x, && (4.7) \\
& \bar{x}(j+1) = A_K\bar{x}(j) + Bc(j),\ j \in \mathbb{I}_{[0,N-1]} && (4.8) \\
& \bar{u}(j) = K\bar{x}(j) + c(j),\ j \in \mathbb{I}_{[0,N-1]} && (4.9) \\
& \bar{x}(j) \in \bar{\mathcal{X}}_j, \quad j \in \mathbb{I}_{[0,N-1]} && (4.10) \\
& \bar{u}(j) \in \bar{\mathcal{U}}_j, \quad j \in \mathbb{I}_{[0,N-1]} && (4.11) \\
& \bar{x}(N) \in \Sigma_0 \ominus \mathcal{R}_N && (4.12)
\end{aligned}$$

where Σ_0 is a polyhedron (Kolmanovsky and Gilbert, 1998), defined as:

$$\Sigma_0 \triangleq \{x : A_K^i x \in \bar{\mathcal{X}}_i, KA_K^i x \in \bar{\mathcal{U}}_i, \text{ for } i \geq 0\}$$

That is, a robust invariant set for the system $x^+ = A_K x + w$ for $w \in \mathcal{W}$ contained in $X_K = \{x \in X : Kx \in U\}$.

The control law is given by

$$k_N^r(x) = Kx(k) + c^0(0; x(k)) \tag{4.13}$$

where $c^0(0; x(k))$ is the first term of the optimal sequence calculated at $x(k)$.

In (Chisci et al., 2001, Theorem 8) is also proved that, given a feasible initial condition $x(0)$, system (4.1) under the control law $u(k) = k_N^r(x(k))$ satisfies that:

- $x(k) \in \mathcal{X}$ and $u(k) \in \mathcal{U}$, for all $k \geq 0$

- $\lim\limits_{k\to\infty} c^0(0; x(k)) = 0$
- $x(k) \to \mathcal{R}_\infty$ as $k \to \infty$

That is, the uncertain system fulfils the constraints for any possible uncertainty, and the control law tends to the linear controller. Then the controlled system is steered to the minimum invariant set.

These results are derived from the demonstration that the sequence

$$\tilde{\mathbf{c}}(x(k+1)) = \{c^0(1; x(k)), \cdots, c^0(N-1; x(k)), 0\} \tag{4.14}$$

is a feasible solution for $x(k+1)$, provided the sequence $\mathbf{c}^0(x(k))$.

In the next section, this control law is extended to the case of tracking target sets.

4.4 Robust MPC for tracking zone regions based on nominal predictions

In this section the proposed controller is presented. The proposed controller is a robust formulation of the MPC for tracking zone regions (Ferramosca et al., 2010a) based on the robust MPC presented in (Chisci et al., 2001).

As in the regulation case, the nominal model of the plant (4.3) is considered to be subject to the restricted constraints:

$$\begin{aligned} \bar{\mathcal{X}}_j &\triangleq \mathcal{X} \ominus \mathcal{R}_j \\ \bar{\mathcal{U}}_j &\triangleq \mathcal{U} \ominus K\mathcal{R}_j \end{aligned} \tag{4.15}$$

As in the nominal case presented in the previous chapter, every nominal steady state and input $z_s = (x_s, u_s)$ is a solution of the equation

$$\begin{bmatrix} A - I_n & B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \mathbf{0}_{n,1} \tag{4.16}$$

Therefore, there exists a matrix $M_\theta \in \mathbb{R}^{(n+m)\times m}$ such that every nominal steady state and input can be posed as

$$z_s = M_\theta \theta \tag{4.17}$$

for certain $\theta \in \mathbb{R}^m$. The subspace of nominal steady outputs is then given by

$$y_s = N_\theta \theta \tag{4.18}$$

where $N_\theta \triangleq [C \ D] M_\theta$.

Defining $\bar{\mathcal{Z}} \triangleq \bar{\mathcal{X}}_N \times \bar{\mathcal{U}}_N$, the set of admissible nominal steady states and inputs and the set of admissible nominal controlled variables are given by

$$\begin{aligned} \bar{\mathcal{Z}}_s &\triangleq \{(x,u) \in \bar{\mathcal{Z}} : (A - I_n)x + Bu = \mathbf{0}_{n,1}\} \\ \bar{\mathcal{Y}}_s &\triangleq \{Cx + Du : \ (x,u) \in \lambda \bar{\mathcal{Z}}_s\} \end{aligned}$$

where $\lambda \in (0,1)$.

The proposed controllers is derived following the results presented in the last section. Firstly, the cost function to minimize is introduced as follows:

$$V_N(x, \Gamma_t; \mathbf{c}, \theta) \triangleq \sum_{j=0}^{N-1} \|c(j)\|_\Psi^2 + V_O(y_s, \Gamma_t) \tag{4.19}$$

where the pair $(x_s, u_s) = M_\theta \theta$ is the artificial steady state and input and $y_s = N_\theta \theta$ the artificial output, all of them parameterized by θ; Γ_t is the zone in which the controlled variables have to be steered. $V_O(y_s, \Gamma_t)$ is the so-called offset cost function and it is such that the following assumption is ensured

Assumption 4.5

1. *Γ_t is a compact convex set.*
2. *$V_O(y_s, \Gamma_t)$ is subdifferential and convex w.r.t. y_s.*
3. *If $y_s \in \Gamma_t$, then $V_O(y_s, \Gamma_t) \geq 0$. Otherwise, $V_O(y_s, \Gamma_t) > 0$.*

In this case, the plant is pre-stabilized by the following control law

$$u(k) = Kx(k) + L\theta + c(k) \tag{4.20}$$

where $L = [-KI]M_\theta$. Then the nominal system can be rewritten as follows:

$$\begin{aligned} \bar{x}^+ &= A_K \bar{x} + BL\theta + Bc \\ \bar{u} &= Kx + L\theta + c \end{aligned} \tag{4.21}$$

The optimization problem $P_N(x, \Gamma_t)$ is now given by:

$$\begin{aligned}
\min_{\mathbf{c},\theta} \quad & V_N(x, \Gamma_t; \mathbf{c}, \theta) \\
s.t. \quad & \bar{x}(0) = x, & (4.22)\\
& \bar{x}(j+1) = A\bar{x}(j) + B\bar{u}(j),\ j \in \mathbb{I}_{[0,N-1]} & (4.23)\\
& \bar{u}(j) = K\bar{x}(j) + L\theta + c(j),\ j \in \mathbb{I}_{[0,N-1]} & (4.24)\\
& \bar{x}(j) \in \bar{\mathcal{X}}_j, \quad j \in \mathbb{I}_{[0,N-1]} & (4.25)\\
& \bar{u}(j) \in \bar{\mathcal{U}}_j, \quad j \in \mathbb{I}_{[0,N-1]} & (4.26)\\
& y_s = N_\theta \theta & (4.27)\\
& (\bar{x}(N), \theta) \in \Omega_t^a & (4.28)
\end{aligned}$$

Notice that the decision variables are: (i) the sequence of the future actions of the nominal system $\mathbf{c}$ and (ii) the parameter vector θ that determines the artificial target steady state, input and output (x_s, u_s, y_s).

Considering the receding horizon policy, the control law is given by

$$\kappa_N(x, \Gamma_t) \triangleq Kx + L\theta^0(x, \Gamma_t) + c^0(0; x, \Gamma_t)$$

where $c^0(0; x, \Gamma_t)$ is the first element of the control sequence $\mathbf{c}^0(x, \Gamma_t)$ which is the optimal solution of problem $P_N(x, \Gamma_t)$. Notice also that, in the following, the optimal value of the cost function will be denoted as $V_N^0(x, \Gamma_t; \mathbf{c}, \theta)$, the optimal value of the other decision variable as $\theta^0(x, \Gamma_t)$, the nominal optimal state trajectory as $\bar{\mathbf{x}}^0(x, \Gamma_t)$ and the optimal artificial reference $(x_s^0(x, \Gamma_t), u_s^0(x, \Gamma_t), y_s^0(x, \Gamma_t))$.

Since the set of constraints of $P_N(x, \Gamma_t)$ does not depend on Γ_t, its feasibility region does not depend on the target region Γ_t. The feasible set of the proposed controller is a polyhedral region $\mathcal{X}_N \subseteq \mathbb{R}^n$ given by the set of initial states that can be steered into $\Omega_t = Proj_x(\Omega_t^a)$ in N steps fulfilling the constraint (4.25), for all admissible disturbances.

4.4.1 Stability of the proposed controller

Consider the following assumption on the controller parameters:

Assumption 4.6

1. *Let $K \in \mathbb{R}^{m\times n}$ be a stabilizing control gain such that the eigenvalues of $(A+BK)$ are in the unit circle.*

2. *Define the extended state $x_a = (x, \theta)$, and*

$$A_a = \begin{bmatrix} A+BK & BL \\ 0 & I_m \end{bmatrix}$$

where $L = [-K\, I_m]M_\theta$. Define also

$$X_a^i = \{(x,\theta) : x \in \bar{\mathcal{X}}_i, Kx + L\theta \in \bar{\mathcal{U}}_i, M_\theta\theta \in \lambda\mathcal{Z}_s\}$$

and

$$\Sigma_t = \{x_a : A_a^i x \in X_a^i, \quad for\ i \geq 0\}$$

Then

$$\Omega_t = \Sigma_t \ominus (\mathcal{R}_N \times \{0\})$$

In the following theorem, stability and constraints satisfaction of the controlled system are stated.

Theorem 4.7 (Stability) *Consider that assumptions 4.1, 4.4, 4.5 and 4.6 hold and consider a given target operation zone Γ_t. The system controlled by the proposed MPC controller $\kappa_N(x,\Gamma_t)$ is such that:*

(i) For all initial condition $x(0) \in \mathcal{X}_N$ and for every Γ_t, the evolution of the system is robustly feasible and admissible, that is, $x(j) \in \mathcal{X}_N$ and $(x(j), \kappa_N(x(j),\Gamma_t)) \in \mathcal{Z}$, $\forall w(k) \in \mathcal{W}$, $k = 0, 1, \cdots, j-1$.

(ii) $\lim_{k\to\infty} c(k) = 0$

(iii) If $\Gamma_t \cap \bar{\mathcal{Y}}_s \neq \emptyset$ then the closed-loop system asymptotically converges to a set $\bar{y}(\infty) \oplus (C+DK)\mathcal{R}_\infty$, such that $\bar{y}(\infty) \in \Gamma_t$.

(iv) If $\Gamma_t \cap \bar{\mathcal{Y}}_s = \emptyset$, the closed-loop system asymptotically converges to a set $y_s^* \oplus (C+DK)\mathcal{R}_\infty$, where y_s^* is the reachable nominal steady output such that

$$y_s^* \triangleq \arg\min_{y_s \in \bar{\mathcal{Y}}_s} V_O(y_s, \Gamma_t)$$

4.5 Stability proof

In this section, the stability proof of theorem 4.7 is presented. Firstly, it is necessary to introduce some lemmas. To this aim, define as $(\mathbf{c}^0(x(k), \Gamma_t), \theta^0(x(k), \Gamma_t))$ the optimal solution of problem $P_N(x, \Gamma_t)$ at the time instant k, where

$$\mathbf{c}^0(x(k), \Gamma_t) = \{c^0(0; x(k), \Gamma_t), c^0(1; x(k), \Gamma_t), ..., c^0(N-1; x(k), \Gamma_t)\}$$

Define the control sequence

$$\tilde{\mathbf{c}}(x(k+1), \Gamma_t) = \{c^0(1; x(k), \Gamma_t), ..., c^0(N-1; x(k), \Gamma_t), 0\}$$

and define $\tilde{\theta}(x(k+1), \Gamma_t) = \theta^0(x(k), \Gamma_t)$. Moreover, define as $\tilde{x}(j; x(k+1), \Gamma_t)$ the j-th step prediction, given $x(k+1)$. Hence

$$\tilde{x}(j; x(k+1), \Gamma_t) = A_K^j x(k+1) + \sum_{i=0}^{j-1} A_K^i B\big[\tilde{c}(j-i-1; x(k+1), \Gamma_t) + L\tilde{\theta}(x(k+1), \Gamma_t)\big]$$

In what follows, the dependence from (x, Γ_t) will be omitted for the sake of clarity, namely, $x(j; k)$ will denote $x(j; x(k), \Gamma_t)$.

Lemma 4.8 *For all $j = 0, ..., N-1$*

$$\tilde{x}(j; k+1) - \bar{x}(j+1; k) = A_K^j w(k)$$

Proof: Since

$$\bar{x}^0(j+1; k) = A_K^j \bar{x}^0(1; k) + \sum_{i=0}^{j-1} A_K^i B\big[c^0(j-i; k) + L\theta^0(k)\big]$$

and

$$\begin{aligned} \tilde{x}(j; k+1) &= A_K^j x(k+1) + \sum_{i=0}^{j-1} A_K^i B\big[\tilde{c}(j-i-1; k+1) + L\tilde{\theta}(k+1)\big] \\ &= A_K^j x(k+1) + \sum_{i=0}^{j-1} A_K^i B\big[c^0(j-i; k) + L\theta^0(k)\big] \end{aligned}$$

hence

$$\begin{aligned} \tilde{x}(j; k+1) - \bar{x}(j+1; k) &= A_K^j\big[x(k+1) - \bar{x}^0(1; k)\big] \\ &= A_K^j w(k) \end{aligned}$$

■

Lemma 4.9 *If $\bar{x}^0(j;k) \in \bar{\mathcal{X}}_j$, then $\tilde{x}(j-1;k+1) \in \bar{\mathcal{X}}_{j-1}$, for all $j = 0, ..., N$.*

Proof: Since $\tilde{x}(j-1;k+1) = \bar{x}^0(j;k) + A_K^{j-1}w(k)$, then

$$\begin{aligned} \tilde{x}(j-1;k+1) \in \bar{\mathcal{X}}_j \oplus A_K^{j-1}\mathcal{W} &= \mathcal{X} \ominus [\bigoplus_{i=0}^{j-1} A_K^i \mathcal{W}] \oplus A_K^{j-1}\mathcal{W} \\ &= \mathcal{X} \ominus [\bigoplus_{i=0}^{j-2} A_K^i \mathcal{W}] \\ &= \bar{\mathcal{X}}_{j-1} \end{aligned}$$

■

Lemma 4.10 *If $K\bar{x}^0(j;k) + c^0(j;k) + L\theta^0(k) \in \bar{\mathcal{U}}_j$, then $K\tilde{x}(j-1;k+1) + \tilde{c}(j-1;k+1) + L\tilde{\theta}(k+1) \in \bar{\mathcal{U}}_{j-1}$, for all $j = 1, ..., N-1$.*

Proof: Taking into account that

$$K\bar{x}^0(j;k) + c^0(j;k) + L\theta^0(k) = K\tilde{x}(j-1;k+1) - KA_K^{j-1}w(k) + \tilde{c}(j-1;k+1) + L\tilde{\theta}(k+1)$$

hence

$$K\tilde{x}(j-1;k+1) + \tilde{c}(j-1;k+1) + L\tilde{\theta}(k+1) \in \bar{\mathcal{U}}_j \oplus A_K^{j-1}\mathcal{W}$$

and

$$\bar{\mathcal{U}}_j \oplus A_K^{j-1}\mathcal{W} = \mathcal{U} \ominus K\mathcal{R}_j \oplus A_K^{j-1}\mathcal{W} = \mathcal{U} \ominus K\mathcal{R}_{j-1} = \bar{\mathcal{U}}_{j-1}$$

■

Lemma 4.11 *[Recursive feasibility of the terminal constraint]For all $k \geq 0$,*

$$(\bar{x}^0(N;k), \theta^0(k)) \in \Omega_t^a$$

Proof: Consider that at time k $(\bar{x}^0(N;k), \theta^0(k)) \in \Omega_t^a$. Since $\Omega_t^a = \Sigma_t \ominus (\mathcal{R}_N \times 0)$, hence

$$(\bar{x}^0(N-1;k+1), \theta^0(k+1)) \in \Sigma_t \ominus (\mathcal{R}_N \times 0) \oplus (A_K^{N-1}\mathcal{W} \times 0)$$

Then, since $(\bar{x}^0(N;k+1),\theta^0(k+1)) = A_a(\bar{x}^0(N-1;k+1),\theta^0(k+1))$, hence

$$(\bar{x}^0(N;k+1),\theta^0(k+1)) \in A_a\Sigma_t \ominus (A_K(\mathcal{R}_N \ominus A_K^{N-1}\mathcal{W}) \times 0)$$

Taking into account that $A_K(\mathcal{R}_N \ominus A_K^{N-1}\mathcal{W}) = \mathcal{R}_N \ominus \mathcal{W}$, then

$$(\bar{x}^0(N;k+1),\theta^0(k+1)) \in A_a\Sigma_t \oplus ((\mathcal{W} \ominus \mathcal{R}_N) \times 0) \subseteq \Sigma_t \ominus (\mathcal{R}_N \times 0) = \Omega_t^a$$

■

4.5.1 Proof of theorem 4.7

In what follows, it will be proved that the closed-loop system is ISS for all $x(0) \in \mathcal{X}_N$.

Proof: From lemmas 4.8, 4.9, 4.10 and 4.11, it is derived that the couple $(\tilde{\mathbf{c}}(k+1),\tilde{\theta}(k+1))$ is a feasible solution of problem $P_N(x,\Gamma_t)$.

Consider now the optimal value of the cost function $V_N^0(x(k),\Gamma_t)$, due to the optimal solution of problem $P_N(x(k),\Gamma_t)$, given by $(\mathbf{c}^0(k),\theta^0(k))$. Define

$$\tilde{V}_N(x(k+1),\Gamma_t;\tilde{\mathbf{c}},\tilde{\theta}) = \sum_{j=0}^{N-1} \|\tilde{c}(j;k+1)\|_\Psi^2 + V_O(y_s,\Gamma_t)$$

Comparing $\tilde{V}_N(x(k+1),\Gamma_t;\tilde{\mathbf{c}},\tilde{\theta})$ with $V_N^0(x(k),\Gamma_t)$, we have that

$$\tilde{V}_N(x(k+1),\Gamma_t;\tilde{\mathbf{c}},\tilde{\theta}) - V_N^0(x(k),\Gamma_t) = -\|c^0(0;k)\|_\Psi^2$$

and hence, by optimality:

$$V_N^0(x(k+1),\Gamma_t) - V_N^0(x(k),\Gamma_t) \leq -\|c^0(0;k)\|_\Psi^2$$

Since $\Psi > 0$, we can state that:

$$\lim_{k\to\infty} c^0(0;k) = 0$$

and (ii) is proved.

The fact that $c^0(0;k) \to 0$ implies that $u(k) \to K(x(k) - x_s^0(k)) + u_s^0(k)$, and hence:

$$x(k) \to x_s^0(k) \oplus \mathcal{R}_\infty, \qquad u(k) \to u_s^0(k) \oplus K\mathcal{R}_\infty$$

Using the same arguments as in the chapter 2, it can be proved that $(x_s^0(k), u_s^0(k))$ converges to the optimal equilibrium point (x_s^*, u_s^*) which is the minimizer of the offset cost function $V_O(y_s, \Gamma_t)$.

Now, the stability of the equilibrium point will be proved. If the uncertainty is null, then (following chapter 2) the system is asymptotically stable in (x_s^*, u_s^*). If $w \neq 0$, the continuity of the control law provides that the closed-loop system is such that the closed-loop prediction $\phi_{cl}(j; x, w) = \phi(j; x, k_N(x, \Gamma_t), w)$ is continuous in x and w. Then, resorting to ISS arguments (Limon et al., 2009a), it can be proved that there exist a $\mathcal{K}L$ function β and a $\mathcal{K}$ function γ such that

$$|x(k) - x_s^*| \leq \beta(|x(0) - x_s^*|, k) + \gamma(\|w\|)$$

for all initial state $x(0) \in \mathcal{X}_N$ and all disturbances $w(k)$. ■

4.6 Properties of the proposed controller

The proposed controller is a robust formulation of the MPC for tracking target sets presented in chapter 3. As a consequence, it inherits all the good properties of that controllers:

- **Steady state optimization** The proposed controller steers the system to a neighborhood of the optimal operating point according to the offset cost function $V_O(y_s, \Gamma_t)$. Then it can be considered that the proposed controller has a steady state optimizer built in and $V_O(y_s, \Gamma_t)$ defines the function to optimize.
- **Feasibility for any reachable target zone** The controller is able to guarantee feasibility for any Γ_t and for any prediction horizon N. Then, it can be derived that the proposed controller is able to lead the system to any admissible target zone (i.e. $\Gamma_t \cap \bar{\mathcal{Y}}_s \neq \emptyset$) even for $N = 1$, if the system starts from an admissible equilibrium point.
- **Changing target zones** Since the set of constraints of $P_N(x, \Gamma_t)$ does not depend on Γ_t, its feasibility region does not depend on the target operating point Γ_t. Therefore, if Γ_t varies with the time, the results of theorem 4.7 still hold.
- **Input target** The proposed controller can be formulated considering input targets of the form $u_{min} \leq u_t \leq u_{max}$, by defining an offset cost function $V_O(u_s, \Gamma_{u,t})$ subdifferential and convex w.r.t. u_s, where $\Gamma_{u,t}$ is a convex polyhedron.

- **Enlargement of the domain of attraction** The fact that the terminal constraint is an invariant set for any equilibrium points makes this set (potentially) larger than the one calculated for regulation to a fixed equilibrium point. Consequently, the domain of attraction of the proposed controller is (potentially) larger than the domain of the standard MPC. This property is particularly interesting for small values of the control horizon.

- **Optimization problem posed as a QP** Since all the ingredients (functions and sets) of the optimization problem $P_N(x, \Gamma_t)$ are convex, then it derives that $P_N(x, \Gamma_t)$ is a convex mathematical programming problem that can be efficiently solved in polynomial time by specialized algorithms (Boyd and Vandenberghe, 2006). As in (Ferramosca et al., 2010a), this problem can be re-casted as a standard QP problem, with a certain choice of the offset cost function. In particular, three formulations allow this recasting:

  (i) distance from a set as ∞-norm
  $$V_O(y_s, \Gamma_t) \triangleq \min_{y \in \Gamma_t} \|y_s - y\|_\infty \tag{4.29}$$
  (ii) distance from a set as 1-norm
  $$V_O(y_s, \Gamma_t) \triangleq \min_{y \in \Gamma_t} \|y_s - y\|_1 \tag{4.30}$$
  (iii) distance from a set as a scaling factor: in this implementation, the target region is defined as
  $$\Gamma_t \triangleq y_t \oplus \Xi_t$$
  where y_t is a desired target point and Ξ_t is a polyhedron that defines the zone. Then
  $$\begin{aligned} V_O(y_s, \Gamma_t) &= \min_{\lambda, y} \lambda \\ s.t. \;\; & \lambda \geq 0 \\ & y - y_t \in \lambda \Xi_t \end{aligned} \tag{4.31}$$

  See (Ferramosca et al., 2010a) or chapter 3 for more details on how to recast the optimization problem $P_N(x, \Gamma_t)$ to obtain a QP problem.

4.7 Illustrative example

In this example, the proposed controller has been tested in a simulation on the 4 tanks process presented in the Appendix A. The objective of this test is to show how the controller maintains the system within some specified zones, rejecting the disturbances applied to the system.

To this aim, in the test, five target zones have been considered. The limits of these zones are given in table. 4.1 Notice that $(h_1^0, h_2^0, h_3^0, h_4^0)$ is the point around which the

Table 4.1: Target zones used in the example

Γ_t	y_{min}	y_{max}
$\Gamma_{t,1}$	$(0.20, 0.20)$	$(0.40, 0.40)$
$\Gamma_{t,2}$	$(1.15, 1.15)$	$(1.35, 1.35)$
$\Gamma_{t,3}$	$(0.30, 0.75)$	$(0.40, 0.85)$
$\Gamma_{t,4}$	$(0.95, 0.75)$	$(1.05, 0.85)$
$\Gamma_{t,5}$	$(h_1^0, h_2^0) - (0.05, 0.05)$	$(h_1^0, h_2^0) + (0.05, 0.05)$

system has been linearized (see Appendix A). Notice also that $\Gamma_{t,3}$ is an unreachable zone for the system.

The initial state is $x_0 = (0.65, 0.65, 0.6658, 0.6242)$. An MPC with $N = 3$ has been considered. The weighting matrices have been chosen as $Q = I_4$ and $R = 0.01 \times I_2$. The disturbances are bounded in the set $\mathcal{W} = \{w : \|w\|_\infty \leq 0.005\}$. The gain matrix K is the given by the LQR, and matrix P is the solution of the Riccati equation. The offset cost function V_O has been chosen as described in (4.31). This choice is motivated by the fact that, if $\mathcal{R}_\infty \subseteq \Xi_t$, then the controller will steer the system into the zone, while in the other case the system will be driven on the boundary of the zone. The system has been discretized using the zero-order hold method with a sampling time of 15 seconds.

Figure 4.1 shows the state-space evolution of the system.

The projection of the invariant set Ω_t onto $\mathcal{Y}$, $\Omega_{t,y}$ is plotted in dashed-dotted line, while the projection of the domain of attraction $\mathcal{Y}_N$, with $N = 3$, is plotted in solid line. The region of admissible steady outputs $\mathcal{Y}_s$ is depicted in dashed line. The blue boxes represent the zones while the light blue sets are the minimum robust invariant set $\mathcal{R}_\infty$ centered in the equilibrium point. The desirable target points y_t, are represented as circles and considered as the center of the target zones. See how the controller steers the system into the zone when is possible and however always into the set $y_s^* \oplus \mathcal{R}_\infty$, as stated in Theorem 4.7.

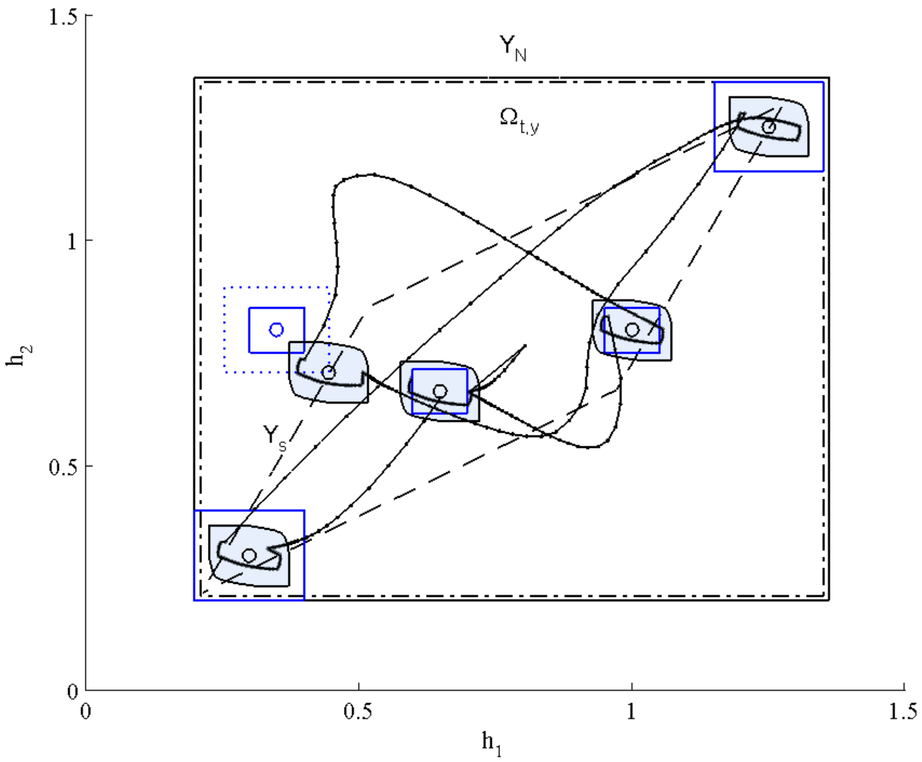


Figure 4.1: State-space evolution of the trajectories.

In Figures 4.2 and 4.3 the time evolution of h_1, h_2, h_3, h_4, q_a and q_b is depicted.

The evolutions of the outputs and the artificial references are drawn respectively in solid and dashed line. The zones are drawn in dotted lines. See how the controller steers the system into the output zone, when is possible, rejecting the disturbances. As already said, $\Gamma_{t,3}$ is not admissible, that is $\Gamma_t \cap \mathcal{Y}_s = \emptyset$. In this case the controller steers the system to a region $y_s^* \oplus \mathcal{R}_\infty$, where y_s^* is the admissible point that minimizes the offset cost function. In the other cases, it can be seen that the controller steers the outputs into the zones. This happens because $\Gamma_t \cap \mathcal{Y}_s \neq \emptyset$ and $y_t \in \mathcal{Y}_s$.
At the sample time $T = 1050$, an impulsive perturbation has been applied to the system. It is clear from figure 4.3 how the controller reacts to this perturbation. As a consequence, in figure 4.2 it is possible to see how, after the perturbation, the system is once again driven into the target zone.

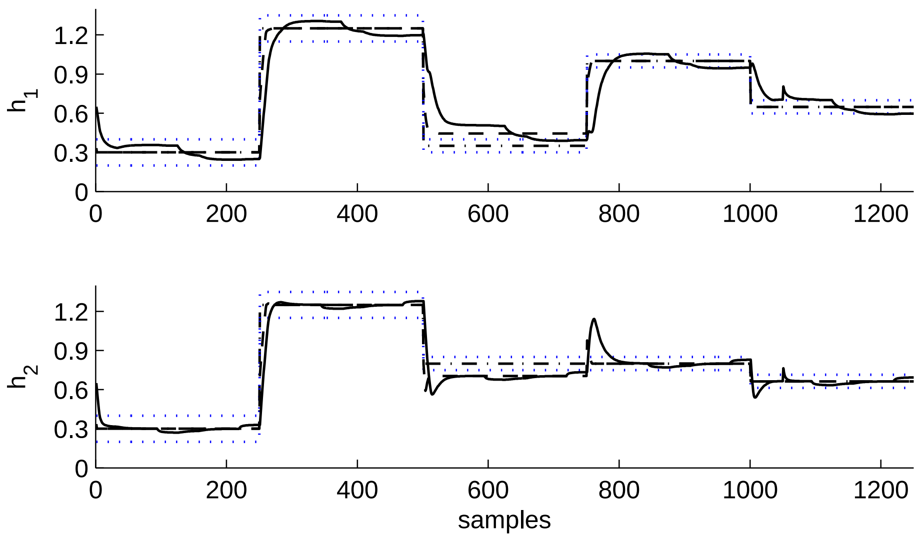


Figure 4.2: Time evolution of h_1 and h_2.

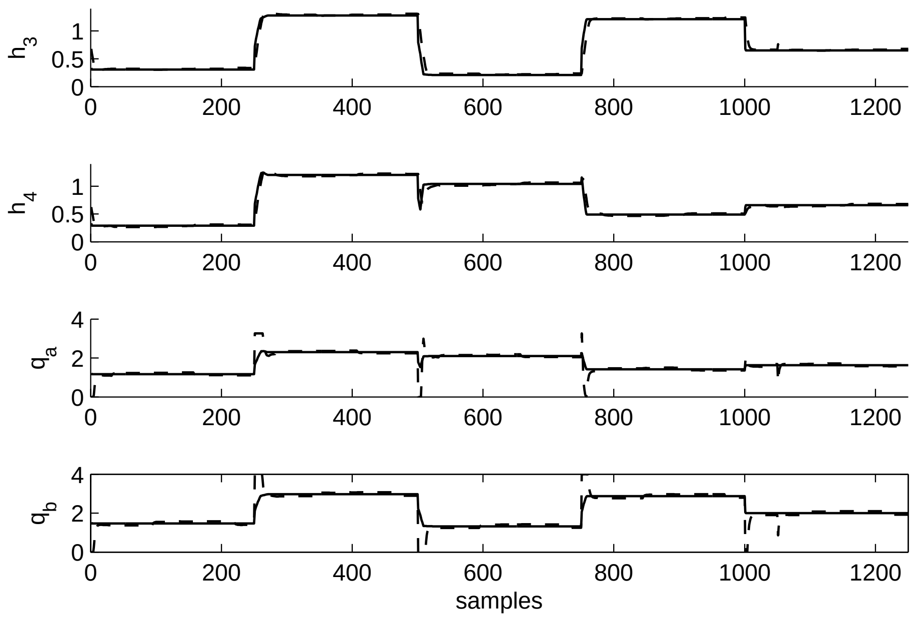


Figure 4.3: Time evolution of h_3, h_4 q_a and q_b.

4.8 Conclusion

The zone control strategy is implemented in applications where the exact values of the controlled outputs are not important, as long as they remain inside a range with specified limits. In this chapter, a robust extension of the MPC for tracking zone regions control has been presented, based on nominal predictions and restricted constraints. From a tracking point of view, the controller considers a set, instead of a point, as target. The concept of deviation between two points used in the offset cost function has been generalized to the concept of distance from a point to a set. A characterization of the offset cost function has been given as the minimal distance between the output and some point inside the target set. The controller ensures recursive feasibility and robust satisfaction of the constraints by using nominal predictions and restricted constraints.

Chapter 5

Distributed MPC for tracking

5.1 Introduction

Large scale control systems usually consist of linked unit of operations and can be divided into a number of subsystems controlled by different agents which may or may not share information. A first approach to this problem is decentralized control, in which interactions between the different subsystems are not considered (Sandell Jr. et al., 1978). The main issue of this solution appears when the intersubsystem interactions become strong. Centralized control, a single agent controls the plantwide system, is another traditional solution that can cope with this control problem. The main problems of this solution are the computational burden and the coordination of subsystems and controller. Distributed control schemes, where agents share open-loop information in order to improve closed-loop performance, solve many of these problems (Rawlings and Mayne, 2009, Chapter 6).
The difference between the distributed control strategies is in the use of this open-loop information. In noncooperative distributed control each subsystem considers the other subsystems information as a known disturbance (Camponogara et al., 2002a; Dunbar, 2007). This strategy leads the whole system to converge to a Nash equilibrium. In cooperative distributed control the agents share a common objective and optimize a cost function that can be considered as the whole system cost function (Venkat, 2006; Pannocchia et al., 2009; Stewart et al., 2010). This strategy is a form of suboptimal control: stability is deduced from suboptimal control theory (Scokaert et al., 1999) and converge to a Pareto optimum is ensured.

MPC is one of the most used control structure to cope with distributed control. In (Magni and Scattolini, 2006) an MPC approach for nonlinear systems is proposed, where no information is exchanged between the local controllers. An input-to-state stability proof for this approach is given in (Raimondo et al., 2007b). In (Liu et al., 2009, 2008) the authors present a controller for networked nonlinear systems, which is based on a Lyapunov-based model predictive control. In (Venkat et al., 2007; Stewart et al., 2010) a cooperative distributed MPC is presented, in which suboptimal input trajectories are used to stabilize the plant.

In this chapter, the MPC for tracking presented in (Limon et al., 2008a) and (Ferramosca et al., 2009a) is extended to the case of large scale distributed systems. Among the different solutions presented in literature, in this chapter we particularly focus our attention on the cooperative formulation for distributed MPC presented in (Rawlings and Mayne, 2009, Chapter 6), in (Venkat, 2006) and in (Stewart et al., 2010). In this formulation, the players share a common objective, which can be considered as the overall plant objective. This means that any player calculates its corresponding inputs by minimizing the same and unique cost function, by means of an iterative (and hence suboptimal) distributed optimization problem. Stability is proved by means of suboptimal MPC theory (Scokaert et al., 1999).

Consider a system described by a linear invariant discrete time model

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, $y \in \mathbb{R}^p$ is the controlled output and x^+ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u})$ where $x = \phi(0; x, \mathbf{u})$. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively. The system is subject to hard constraints on state and control:

$$x(k) \in X, \quad u(k) \in U \tag{5.2}$$

for all $k \geq 0$. $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are compact convex polyhedra containing the origin in their interior. It is assumed that the following hypothesis hold.

Assumption 5.1 *The pair (A,B) is stabilizable and the state is measured at each sampling time.*

5.1.1 Characterization of the equilibrium points of the plant

The steady state, input and output of the plant (x_s, u_s, y_s) are such that (5.1) is fulfilled, i.e. $x_s = Ax_s + Bu_s$, and $y_s = Cx_s + Du_s$.

We define the sets of admissible equilibrium states, inputs and outputs as

$$\mathcal{Z}_s = \{(x,u) \in X \times U \mid x = Ax + Bu\} \tag{5.3}$$

$$\mathcal{X}_s = \{x \in X \mid \exists u \in U \text{ such that } (x,u) \in \mathcal{Z}_s\} \tag{5.4}$$

$$\mathcal{Y}_s = \{y = C_x + D_u \mid (x,u) \in \lambda \mathcal{Z}_s\} \tag{5.5}$$

Notice that $\mathcal{X}_s$ is the projection of $\mathcal{Z}_s$ onto X.

The steady conditions of the system can be determined by a suitable parametrization. In (Limon et al., 2008a) the authors state that the steady state and input (x_s, u_s) of the system can be parameterized as a linear combination of a vector $\theta \in \mathbb{R}^m$. In order to present the results of this paper in a more intuitive way, we choose a steady output y_s to parameterize every equilibrium point (x_s, u_s). This parametrization is possible if and only if *Lemma 1.14* in (Rawlings and Mayne, 2009, p. 83) holds. If this condition does not hold, the parametrization presented in chapter 2 has to be used.

Under assumption 5.1 and *Lemma 1.14* in (Rawlings and Mayne, 2009, p. 83), any steady state and input of system (5.1) associated to this y_s, namely, every solution of the following equation,

$$\begin{bmatrix} A - I_n & B & \mathbf{0}_{p,1} \\ C & D & -I_p \end{bmatrix} \begin{bmatrix} x_s \\ u_s \\ y_s \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n,1} \\ \mathbf{0}_{p,1} \end{bmatrix} \tag{5.6}$$

is given by $(x_s, u_s) = M_y y_s$, where M_y is a suitable matrix.

5.1.2 Distributed model of the plant

In this chapter, a distributed control framework is considered based on a suitable partition of the plant into a collection of coupled subsystems. In virtue of (Stewart et al., 2010, Section 3.1.1) and (Rawlings and Mayne, 2009, Chapter 6, pp. 421-422), we consider that the plant given by (5.1) is partitioned in M subsystems (where $M \leq m$) modeled as follows:

$$\begin{aligned} x_i^+ &= A_i x_i + \sum_{j=1}^{M} \bar{B}_{ij} u_j \\ y_i &= C_i x_i + \sum_{j=1}^{M} \bar{D}_{ij} u_j \end{aligned} \tag{5.7}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_j \in \mathbb{R}^{m_j}$, $y_i \in \mathbb{R}_i^p$, $A_i \in \mathbb{R}^{n_i \times n_i}$ and $B_{ij} \in \mathbb{R}^{n_i \times m_j}$. Without loss of generality, it is considered that $u = (u_1, \cdots, u_M)$.

As proved in (Stewart et al., 2010), any plant can be partitioned as proposed for a certain definition of x_i. If the couple (C_i, A_i) is observable, the inner state of the partition can be calculated or estimated from the measured output of the subsystem y_i.

For the sake of simplicity of the exposition, the results will be presented for the case of two players game. In this case, the plant can be represented in the form:

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ &= \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix} u_2 \\
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{D}_{11} \\ \bar{D}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \bar{D}_{12} \\ \bar{D}_{22} \end{bmatrix} u_2
\end{aligned}$$

The solution of this system, given the sequences of control inputs $\mathbf{u}_1$ and $\mathbf{u}_2$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u}_1, \mathbf{u}_2)$ where $x = \phi(0; x, \mathbf{u}_1, \mathbf{u}_2)$.

The problem we consider is the design of a cooperative distributed MPC controller to track a (possible time-varying) plant-wide target output $y_t = (y_{t,1}, y_{t,2})$. The proposed distributed controller will ensure convergence to the target if this is admissible or as close as possible if not admissible. This control law is shown in the following section.

5.2 Cooperative MPC

Among the existing solutions for the distributed predictive control problem, we focus our attention on the cooperative game (Stewart et al., 2010; Rawlings and Mayne, 2009, Chapter 6, p. 433). In this case, the two players share a common (and hence coupled) objective, which can be considered as the overall plant objective.

$$V_N^c(x, y_t; \mathbf{u}) \;=\; \sum_{j=0}^{N-1} \|x(j) - x_t\|_Q^2 + \|u(j) - u_t\|_R^2 + \|x(N) - x_t\|_P^2$$

where $x = (x_1, x_2)$, $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ and (x_t, u_t, y_t) defines the state, input and output of the target, that it is assumed to be an equilibrium point of the centralized model of the plant.

In cooperative distributed MPC, each i-th agent calculates its corresponding input u_i by solving an iterative decentralized optimization problem. The solution of the i-th agent at the iteration p will be denoted as $\mathbf{u}_i^{[p]}$. Based on this, the solution of each agent at the next iteration $p+1$ is calculated from the solution of the optimization problem $P_i^c(x, y_t, \mathbf{u}^{[p]})$ for the i-th agent, that depends on the state $x = (x_1, x_2)$, the target y_t and the solution of the p-th iteration $\mathbf{u}^{[p]} = (\mathbf{u}_1^{[p]}, \mathbf{u}_2^{[p]})$. The optimization problem $P_i^c(x, y_t, \mathbf{u}^{[p]})$ is given by:

$$\mathbf{u}_i^0 = \arg\min_{\mathbf{u}_i} V_N^c(x, y_t; \mathbf{u}) \tag{5.8a}$$

$$s.t. \tag{5.8b}$$

$$x_q(j+1) = A_q x_q(j) + \sum_{\ell=1}^{2} B_{q\ell} u_\ell(j), \ q \in \mathbb{I}_{1,2} \tag{5.8c}$$

$$x_1(0) = x_1, \ x_2(0) = x_2 \tag{5.8d}$$

$$(\mathbf{u}_1^{[p]}, \mathbf{u}_2^{[p]}) = \mathbf{u}^{[p]}, \tag{5.8e}$$

$$u_\ell(j) = u_\ell^{[p]}(j) \quad \ell \in \mathbb{I}_{1,2} \setminus i, \tag{5.8f}$$

$$(x_1(j), x_2(j)) \in X, \tag{5.8g}$$

$$(u_1(j), u_2(j)) \in U, \quad j = 0, ..., N-1 \tag{5.8h}$$

$$(x_1(N), x_2(N)) = x_t \tag{5.8i}$$

Denoting the optimal solution of this problem as $\mathbf{u}_i^0$, the solution at the current iteration $p+1$ will be given by

$$\mathbf{u}_1^{[p+1]} = w_1 \mathbf{u}_1^0 + w_2 \mathbf{u}_1^{[p]} \tag{5.9a}$$

$$\mathbf{u}_2^{[p+1]} = w_1 \mathbf{u}_2^{[p]} + w_2 \mathbf{u}_2^0 \tag{5.9b}$$

$$w_1 + w_2 = 1 \quad w_1, w_2 > 0$$

At time k, the iterative method finishes at the iteration $\bar{p}$, once the computation time is expired or a given accuracy of the solution is achieved. Then the best available solution $u_1 = u_1^{[\bar{p}]}(0)$ and $u_2 = u_2^{[\bar{p}]}(0)$ is applied to the plant. Hence, the overall predictive controller can be considered as a suboptimal MPC since the distributed solution is a suboptimal solution of the centralized MPC problem.

Based on the stability theory of suboptimal MPC, it has been demonstrated that this decentralized approach ensures recursive feasibility, optimality (in case of uncoupled constraints) and asymptotic stability under mild assumptions. See (Rawlings and Mayne, 2009, Chapter 6, pp. 446-453) and (Stewart et al., 2010) for a more detailed exposition.

If the setpoint of the controller $(y_{t,1}, y_{t,2})$ is changed, then the corresponding equilibrium point of the optimization problem (x_t, u_t, y_t) must be recalculated solving a target problem. In distributed MPC this target problem is typically solved in a distributed way in such a way that there is a target problem for each agent (Rawlings and Mayne, 2009, Section 6.3.4). If the constraints of each subsystem are decoupled, then the distributed target problem ensures that the distributed controller steers the system to the calculated target. But if the constraints of the problem are coupled, then the optimality of the target problem might be lost, and the controller might fail to steer the plant to the desired target. In this case, it is recommended to use the centralized approach to solve the target problem.

On the other hand, for a centralized or a distributed target problem solution, the distributed controller may become infeasible due to the change in the setpoint, leading to the necessity of redesigning the controller.

In this chapter, a new cooperative distributed MPC for tracking is presented aimed to ensure convergence to the centralized optimal target solution and guaranteeing feasibility after any change of the setpoint of the plant.

5.3 Cooperative MPC for tracking

The distributed control scheme proposed in this chapter extends the MPC for tracking presented in (Limon et al., 2008a; Ferramosca et al., 2009a) to a cooperative distributed framework. As in the centralized case, an artificial equilibrium point of the plant (x_s, u_s, y_s), characterized by y_s, is added as decision variable and the following modified cost function is considered:

$$V_N(x, y_t; \mathbf{u}, y_s) = \sum_{j=0}^{N-1} \|x(j)-x_s\|_Q^2 + \|u(j)-u_s\|_R^2 + \|x(N)-x_s\|_P^2 + V_O(y_s, y_t)$$

where $x = (x_1, x_2)$, $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ and (x_s, u_s, y_s) is the artificial equilibrium point of the plant given by y_s. The function $V_O(y_s, y_t)$ is the so called *offset cost function* and it is defined as follows:

Definition 5.2 *Let $V_O(y_s, y_t)$ be a convex and positive definite function in y_s such that the minimizer of*

$$\min_{y_s \in \mathcal{Y}_s} V_O(y_s, y_t)$$

is unique.

This function $V_O(y_s, y_t)$ is a measure of the (economic) cost associated to a given setpoint y_s. This function is typically chosen as a function of the distance $\|y_s - y_t\|$ (Ferramosca et al., 2009a).

The following assumptions are considered to prove stability of the controller:

Assumption 5.3

1. *Let $R \in \mathbb{R}^{m\times m}$ be a positive semidefinite matrix and $Q \in \mathbb{R}^{n\times n}$ a positive semi-definite matrix such that the pair $(Q^{1/2}, A)$ is observable.*

2. *Let $K \in \mathbb{R}^{m\times n}$ be a stabilizing control gain for the centralized system, such that $(A + BK)$ has all the eigenvalues in the unit cirle.*

3. *Let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix for the centralized system such that:*

$$(A+BK)'P(A+BK)-P=-(Q+K'RK) \tag{5.10}$$

4. *Let $\Omega_t^a \subseteq \mathbb{R}^{n+p}$ be an admissible polyhedral invariant set for tracking for system* (5.1) *subject to* (5.2)*, for a given gain K (Limon et al., 2008a).*
   That is, given the extended state $a = (x, y_s)$, for all $a \in \Omega_\lambda$, then $a^+ = A_a a \in \Omega_t^a$, where A_a is the closed-loop matrix given by

$$A_a = \begin{bmatrix} A + BK & BL \\ 0 & I_p \end{bmatrix}$$

   and $L = [-K,\ I_p]M_y$. Furthermore Ω_t^a must be contained in the polyhedral set W_λ given by

$$W_\lambda = \{(x, y_s) \in X \times \mathcal{Y}_s : Kx + Ly_s \in U\}$$

As in the regulation case, the control action to be applied at each sampling time is calculated by an iterative method where an optimization problem for each agent is solved at each iteration. The optimization problem that each i-th agent solves at the

$p+1$ iteration is denoted as $P_i(x, y_t, \mathbf{u}^{[p]})$ and it is given by:

$$
\begin{aligned}
(\mathbf{u}_i^0, y_{s,i}^0) \quad &= \quad \arg\min_{\mathbf{u}_i, y_s} V_N(x, y_t; \mathbf{u}, y_s) && (5.11a)\\
s.t. & && (5.11b)\\
& x_q(j+1) = A_q x_q(j) + \sum_{\ell=1}^{2} B_{q\ell} u_\ell(j),\ q \in \mathbb{I}_{1,2} && (5.11c)\\
& x_1(0) = x_1,\ x_2(0) = x_2 && (5.11d)\\
& (\mathbf{u}_1^{[p]}, \mathbf{u}_2^{[p]}) = \mathbf{u}^{[p]}, && (5.11e)\\
& u_\ell(j) = u_\ell^{[p]}(j) \quad \ell \in \mathbb{I}_{1,2} \setminus i, && (5.11f)\\
& (x_1(j), x_2(j)) \in X, && (5.11g)\\
& (u_1(j), u_2(j)) \in U, \quad j = 0, ..., N-1 && (5.11h)\\
& (x(N), y_s) \in \Omega_t^a && (5.11i)
\end{aligned}
$$

Based on the solution of this optimization problem for each agent, namely $\mathbf{u}_1^0$ and $\mathbf{u}_2^0$, the solution of the $p+1$-iteration is given by

$$
\begin{aligned}
\mathbf{u}_1^{[p+1]} &= w_1 \mathbf{u}_1^0 + w_2 \mathbf{u}_1^{[p]} && (5.12a)\\
\mathbf{u}_2^{[p+1]} &= w_1 \mathbf{u}_2^{[p]} + w_2 \mathbf{u}_2^0 && (5.12b)\\
y_s^{[p+1]} &= w_1 y_{s,1}^0 + w_2 y_{s,2}^0 && (5.12c)\\
w_1 + w_2 &= 1 \quad w_1, w_2 > 0
\end{aligned}
$$

As in (Stewart et al., 2010), once the algorithm reaches the last iteration $\bar{p}$, the inputs of the plant are $u_1(k) = u_1^{[\bar{p}]}(0;k)$ and $u_2(k) = u_2^{[\bar{p}]}(0;k)$.

To proceed with the analysis of the proposed controller, we will denote

$$\mathbf{v} = (\mathbf{u}_1, \mathbf{u}_2, y_s)$$

$\mathbf{v}$ is said to be feasible at x if each optimization problem $P_i(x, y_t, (\mathbf{u}_1, \mathbf{u}_2))$ is feasible for all $i \in \mathbb{I}_{1:2}$. The set of states for which there exists a feasible $\mathbf{v}$ is denoted as $\mathcal{X}_N$. Notice that this set is equal to the feasible set of the centralized MPC for tracking (Limon et al., 2008a), i.e. the set of states that can be admissibly steered to $Proj_x(\Omega_t^a)$ in N steps. Besides, we will denote $V_N(x, yt, \mathbf{v}) = V_N(x, y_t, (\mathbf{u}_1, \mathbf{u}_2))$.

In order to define precisely the proposed cooperative control scheme, the initial solution $\mathbf{v}^{[0]}$ of the iterative procedure (5.12) must be defined. Since the proposed distributed MPC can be considered as a suboptimal formulation of the centralized MPC, this initialization plays the role of the *warm start* of the suboptimal MPC and determines recursive feasibility and convergence of the control algorithm. In cooperative MPC for regulation (Stewart et al., 2010; Rawlings and Mayne, 2009), the *warm start* control sequence is obtained by discarding the first input, shifting the rest of the sequence forward one step and setting the last input to steady input of the target. In this paper we propose the following algorithm:

Algorithm 5.4

Given the solution $\mathbf{v}(k)$, *the objective is to calculate the warm start at sampling time* $k+1$, *denoted as*

$$\mathbf{v}(k+1)^{[0]} = (\mathbf{u}_1(k+1)^{[0]}, \mathbf{u}_1(k+1)^{[0]}, y_s(k+1)^{[0]}).$$

1. *Define the first candidate initial solution:*

$$\begin{aligned}\tilde{\mathbf{u}}_1(k{+}1) &= \{u_1(1;k), ..., u_1(N-1;k), u_{c,1}(N)\}\\ \tilde{\mathbf{u}}_2(k{+}1) &= \{u_2(1;k), ..., u_2(N-1;k), u_{c,2}(N)\}\end{aligned}$$

where

$$u_c(N) = (u_{c,1}(N), u_{c,2}(N)) = Kx(N) + Ly_s^0(k)$$

is the centralized solution given by the centralized terminal control law, and $x(N) = \phi(N; x(k), \mathbf{u}_1(k), \mathbf{u}_2(k))$.

2. *Define the second candidate initial solution:*

$$\begin{aligned}\hat{\mathbf{u}}_1(k{+}1) &= \{\hat{u}_{c,1}(0), ..., \hat{u}_{c,1}(N-1)\}\\ \hat{\mathbf{u}}_2(k{+}1) &= \{\hat{u}_{c,2}(0), ..., \hat{u}_{c,2}(N{-}1)\}\end{aligned}$$

where $(\hat{u}_{c,1}(j), \hat{u}_{c,2}(j)) = \hat{u}_c(j)$ *and*

$$\begin{aligned}\hat{x}(0) &= x(k+1)\\ \hat{x}(j+1) &= (A+BK)\hat{x}(j) + BLy_s^0(k),\ j \in \mathbb{I}_{1:N-2}\\ \hat{u}_c(j) &= K\hat{x}(j) + Ly_s^0(k)\end{aligned}$$

3. ***IF*** $(x(k+1), y_s^0(k)) \in \Omega_t^a$ ***AND*** $V_N(x(k+1), y_t, \hat{\mathbf{u}}) \leq V_N(x(k+1), y_t, \tilde{\mathbf{u}})$
SET

$$\mathbf{v}(k+1)^{[0]} = (\hat{\mathbf{u}}_1(k+1), \hat{\mathbf{u}}_2(k+1), y_s^0(k))$$

ELSE

$$\mathbf{v}(k+1)^{[0]} = (\tilde{\mathbf{u}}_1(k+1), \tilde{\mathbf{u}}_2(k+1), y_s^0(k))$$

As usual in the suboptimal MPC optimization algorithm, the proposed *warm start* for the first optimization iteration $p = 0$ is given by the previous optimal sequence, shifted of one position, with the last control move given by the centralized terminal control law applied to the predicted terminal state of the overall plant and the same artificial steady output, that is $(\tilde{\mathbf{u}}_1(k+1), \tilde{\mathbf{u}}_2(k+1), y_s(k))$. But, according to the algorithm, when the state of the system reaches the invariant set for tracking, that is $(x(k+1), y_s^0(k)) \in \Omega_t^a$, it is desirable that the distributed MPC achieves a *better* cost than cost of using the centralized terminal controller. If this is not possible, that is $V_N(x(k+1), y_t, \hat{\mathbf{u}}, y_s(k)) \leq V_N(x(x+1), y_t, \tilde{\mathbf{u}}, y_s(k))$, hence the centralized terminal control law is chosen as *warm start*. With this choice convergence, to the optimal centralized target and controllability of the solution are ensured.

Remark 5.5 *In case of terminal equality constraint, that is, $\Omega_t^a = \mathcal{X}_s \times \mathcal{Y}_s$, this algorithm can be used taking as terminal controller the dead-beat controller, as matrix P the corresponding solution of* (5.10) *and $N \geq n$.*

At each sampling time k, the initial *warm start* $\mathbf{v}^{[0]}(k)$ is calculated using algorithm 5.4 and then, $\mathbf{v}^{[p]}(k)$ is obtained from the iterative procedure given by (5.11) and (5.12). At a certain number of iteration $\bar{p}$, the final solution, denoted as

$$\mathbf{v}(k) = (\mathbf{u}_1^{[\bar{p}]}(k), \mathbf{u}_2^{[\bar{p}]}(k), \hat{y}_s^{[\bar{p}]}(k)),$$

is achieved. This solution is a function of: (i) the current state $x(k)$ and (ii) the initial feasible solution $\mathbf{v}^{[0]}(k)$ that depends on $\mathbf{v}(k-1)$. Then, following (Stewart et al., 2010), the overall control law can be posed as

$$\mathbf{v}(k+1) = g(x(k), \mathbf{v}(k)) \tag{5.13a}$$

$$x(k+1) = Ax(k) + BH\mathbf{v}(k) \tag{5.13b}$$

where g is a suitable function and H is an appropriate constant matrix.

The stabilizing properties of this controller are stated in the following theorem.

Theorem 5.6 *[Asymptotic stability] Consider that the assumptions 5.1 and 5.3 hold. Let $\mathcal{X}_N$ be the feasible set of states of problem* (5.11)*. Then for all $x(0) \in \mathcal{X}_N$ and for all y_t, the closed-loop system is asymptotic stable and converges to an equilibrium point $(x_s^*, u_s^*) = M_y y_s^*$ such that*

$$y_s^* = \arg \min_{y_s \in \mathcal{Y}_s} V_O(y_s, y_t)$$

Moreover, if $y_t \in \mathcal{Y}_s$, then $y_s^* = y_t$.

5.4 Properties of the proposed controller

The proposed controller provides the following properties to the closed-loop system:

- As in the centralized case (see chapter 2), the domain of attraction of the proposed controller is (potentially) larger than the domain of the standard distributed MPC, since this set is defined for any equilibrium point.
- The proposed controller is able to track any changing setpoint, maintaining the recursive feasibility and constraint satisfaction, since the optimization problem is feasible for any y_t.
- In cooperative MPC, the target problem solved in a distributed way, converges to the centralized optimum only if the constraints are uncoupled. In case case of coupled constraints, it is recommended to use the centralized approach to solve the target problem (Rawlings and Mayne, 2009, Section 6.3.4).
  The proposed controller ensures convergence to the centralized optimal equilibrium point, since every agent solves an optimization problem with a centralized offset cost function. Remarkably, this property holds for any suboptimal solution provided by the controller due, for instance, to the effect of coupled constraints between agents, or to a small number of iterations $\bar{p}$. Furthermore, this equilibrium point is the admissible equilibrium which minimizes the offset cost function.

5.5 Stability proof

In this section, the proof of Theorem 5.6 and the lemmas necessary to this proof, are presented.

Lemma 5.7 *[Recursive feasibility] Given a feasible initial solution* $\mathbf{v}^{[0]}(k)$*, the solution* $\mathbf{v}^{[p]}(k)$ *is feasible* $\forall p \geq 0$ *and* $k \geq 0$*,.*

Proof:

- Recursive feasibility of the iteration p.

Consider $k = 0$ and $p = 0$. Since U and Ω_t^a are convex sets and the two triples $(\mathbf{u}_1^0(x, y_t, \mathbf{v}^{[0]}), \mathbf{u}_2^{[0]}, y_{s,1}^{[0]}(x, y_t, \mathbf{v}^{[0]}))$ and $(\mathbf{u}_1^{[0]}, \mathbf{u}_2^0(x, y_t, \mathbf{v}^{[0]}), y_{s,2}^{[0]}(x, y_t, \mathbf{v}^{[0]}))$ are feasible solutions, hence the convex combination of these solutions is also feasible. Similarly, this is proved for any $p \geq 1$.

- Recursive feasibility of the time instant k.

  Consider that the solution $\mathbf{v}(k)$ is achieved and $\mathbf{v}^{[0]}(k+1)$ is calculated by the algorithm 5.4. If $(x(k+1), y_s^0(k)) \in \Omega_t^a$, and $V_N(x(k+1), y_t, \hat{\mathbf{u}}) \leq V_N(x(k+1), y_t, \tilde{\mathbf{u}})$, then $\mathbf{v}^{[0]}(k+1)$ is feasible since the centralized terminal control law provides a feasible solution. Otherwise, the standard shifted solution is used, which is feasible thanks to the feasibility of the terminal controller.

■

Lemma 5.8 *[Convergence of the algorithm] For any $k \geq 0$ and $p \geq 0$, the obtained cost function is such that*

$$V_N(x(k), y_t, \mathbf{v}^{[p+1]}(k)) \leq V_N(x(k), y_t, \mathbf{v}^{[p]}(k))$$

Proof:

In this proof, the time dependence has been removed for the sake of simplicity. Given the solution $\mathbf{v}^{[p]}(k)$, the following two solutions are computed

$$\begin{aligned} \mathbf{v}_a &= (\mathbf{u}_1^0(x, y_t, \mathbf{u}^{[p]}), \mathbf{u}_2^{[p]}, y_s^{[p]}) \\ \mathbf{v}_b &= (\mathbf{u}_1^{[p]}, \mathbf{u}_2^0(x, y_t, \mathbf{u}^{[p]}), y_s^{[p]}) \end{aligned}$$

From the definition of $P_i(x, y_t, \mathbf{u}^{[p]})$, these two solutions are feasible for this optimization problem. Besides both solutions provide a lower cost than $\mathbf{v}^{[p]}$.
Then from convexity of the optimal cost function and the fact that $\mathbf{v}^{[p+1]}$ is the optimal solution of $P_i(x, y_t, \mathbf{u}^{[p]})$, we have that

$$\begin{aligned} V_N(x, y_t, \mathbf{v}^{[p+1]}) &\leq w_1 V_N(x, y_t, \mathbf{v}_a) + w_2 V_N(x, y_t, \mathbf{v}_b) \\ &\leq w_1 V_N(x, y_t, \mathbf{v}^{[p]}) + w_2 V_N(x, y_t, \mathbf{v}_b) \\ &= V_N(x, y_t, \mathbf{v}^{[p]}) \end{aligned}$$

■

Lemma 5.9 *[Local bounding] Let k be an instant such that $(x(k), y_s^{[0]}(k)) \in \Omega_t^a$. Then*

$$V_N(x(k), y_t; \mathbf{u}_1^{[p]}(k), \mathbf{u}_2^{[p]}(k), y_s^{[p]}(k)) \leq \|x(k) - x_s^{[0]}(k)\|_P^2 + V_O(y_s^{[0]}(k), y_t)$$

Proof: If $(x(k), y_s^{[0]}(k)) \in \Omega_t^a$, hence the initialization of the algorithm ensures that

$$V_N(x(k), y_t; \mathbf{u}_1^{[0]}(k), \mathbf{u}_2^{[0]}(k), y_s^{[0]}(k)) \leq \|x(k) - x_s^{[0]}(k)\|_P^2 + V_O(y_s^{[0]}(k), y_t)$$

This fact and lemma 5.8 prove the lemma. ■

Lemma 5.10 *Let y_∞ and a time instant k such that, $(x_\infty, u^+(j;k)) = M_y y_\infty$ and $y_s(k) = y_\infty$. Then, $V_N^0(x_\infty, y_t) = V_O(y_\infty, y_t)$.*

Proof: It is clear that y_∞ is a fixed point for the closed-loop system. At time k: $x(k) = x_\infty$, $u(k) = u_\infty$, $y_s(k) = y_\infty$. This implies that $x(k+1) = x_\infty$ and $y_s^{[0]}(k+1) = y_\infty$, and hence $(x(k+1), y_s^{[0]}(k+1)) \in \Omega_t^a$. Hence

$$\begin{aligned} V_N^0(x(k+1), y_t) &\leq V_N(x(k+1), y_t; \mathbf{u}_1^{[0]}(k+1), \mathbf{u}_2^{[0]}(k+1), y_s^{[0]}(k+1)) \\ &\leq \|x(k+1) - x_s^{[0]}(k+1)\|_P^2 + V_O(y_s^{[0]}, y_t) \end{aligned}$$

Since $x(k+1) = x_\infty$ and $y_\infty = y_s^{[0]}(k+1)$, hence $V_N^0(x_\infty; y_t) \leq V_0(y_\infty, y_t)$.
Let $V_{N,c}^0(x_\infty, y_t)$ be the optimal centralized solution taking $y_s = y_\infty$. Then, $V_{N,c}^0(x_\infty, y_t) = V_O(y_\infty, y_t)$. Hence, since y_∞ is a fixed point, $y_s^0(k+1) = y_\infty$. Therefore, $(\mathbf{u}_1^0(k+1), \mathbf{u}_2^0(k+1))$ is a suboptimal solution of the centralized problem, and hence

$$V_N^0(x(k+1), y_t) \geq V_{N,c}^0(x(k+1), y_t) = V_{N,c}^0(x_\infty) = V_O(y_\infty, y_t)$$

which proves the lemma. ■

5.5.1 Proof of theorem 5.6

Proof: Given the initial solution in $x(0)$, $\mathbf{u}_1^{[0]}(0)$ and $\mathbf{u}_2^{[0]}(0)$, lemma 5.7 ensures that $\mathbf{u}_1^0(k)$ and $\mathbf{u}_2^0(k)$ are admissible and moreover $x(k) \in X$ for any k.
From lemma 5.8:

$$\begin{aligned} V_N^0(x(k+1), y_t) - V_N^0(x(k), y_t) &\leq V_N(x(k+1), y_t; \mathbf{u}_1^{[0]}(k+1), \mathbf{u}_2^{[0]}(k+1), y_s^{[0]}(k+1)) - V_N^0(x(k), y_t) \\ &\leq V_N(x(k+1), y_t; \tilde{\mathbf{u}}_1(k+1), \tilde{\mathbf{u}}_2(k+1), y_s^0(k)) - V_N^0(x(k), y_t) \\ &\leq -\|x(k) - x_s^0(k)\|_Q^2 - \|u(k) - u_s^0(k)\|_R^2 \end{aligned}$$

Given that the cost function is a positive definite function:

$$\lim_{k\to\infty}|x(k)-x_\infty|=0,\ \lim_{k\to\infty}|u(k)-u_\infty|=0,\ \lim_{k\to\infty}|y_s(k)-y_\infty|=0$$

and $(x_\infty, u_\infty) = M_y y_\infty$. By continuity, we can state that the system converges to a fixed point.
Hence, using lemma 5.10, $V_N^0(x_\infty, y_t) = V_O(y_\infty, y_t)$. Using same arguments as in (Ferramosca et al., 2009a, Theorem 1), we can state that y_∞ is the minimizer of $V_O(y_s, y_t)$, that is $y_\infty = y_s^*$ and then $x_\infty = x_s^* \in X$ and $u_\infty = u_s^* \in U$. Moreover, if $y_t \in \mathcal{Y}_s$, then $y_s^* = y_t$.

Finally, the fact that (x_s^*, u_s^*) is a stable equilibrium point for the closed-loop system is proved. That is, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $|x(0) - x_s^*| \leq \delta$, then $|x(k) - x_s^*| \leq \varepsilon$. Notice that the region $\mathcal{B} = \{x : |x(k) - x_s^*| \leq \varepsilon\} \subseteq \mathcal{X}_N$ and this is true because $x_s^* \in int(\mathcal{X}_N)$.

Hence, define the function $W(x, y_t) = V_N^0(x, y_t) - V_O(y_s^*, y_t)$. Then, $W(x_s^*, y_t) = 0$. This function is such that $\alpha_W(|x - x_s^*|) \leq W(x, y_t) \leq \beta_W(|x - x_s^*|)$, where α_W and β_W are suitable $\mathcal{K}_\infty$ functions. In fact:

- $W(x, y_t) \geq \alpha_l(|x - x_s^0|) + \alpha_O(|x_s^0 - x_s^*|)$. This comes from the fact that the stage cost function is a positive definite function and from the definition of V_O. Then

$$\begin{aligned} W(x, y_t) &\geq \alpha_W(|x - x_s^0| + |x_s^0 - x_s^*|) \\ &\geq \alpha_W(|x - x_s^*|) \end{aligned}$$

- Notice that

$$V_N^0(x, y_t) \leq V_N(x, y_t; \mathbf{v}^{[0]}(k)) \leq |x - x_s^*|_P^2 + V_O(y_s^*, y_t) \leq \beta_W(|x - x_s^*|)$$

  Hence $W(x, y_t) \leq \beta_W(|x - x_s^*|)$.

Then, $\alpha_W(|x(k) - x_s^*|) \leq W(x(k), y_t) \leq W(x(0), y_t) \leq \beta_W(|x - x_s^*|)$ and, hence, $|x(k) - x_s^*| \leq \alpha_W^{-1} \circ \beta_W(|x(0) - x_s^*|)$. So, picking $\delta = \beta_W^{-1} \circ \alpha_W(\varepsilon)$, then $|x(k) - x_s^*| \leq \alpha_W^{-1} \circ \beta_W(\delta) \leq \varepsilon$, proving the stability of x_s^*. ■

5.6 Illustrative example

In this section, an example to test the performance of the proposed controller, is presented. The system adopted is the 4 tanks process presented in the Appendix A.

5.6.1 Distributed model

In order to test de cooperative distributed MPC for tracking presented in the paper, the linearized model presented in the Appendix A has been partitioned in two subsystems in such a way that the two subsystems are interconnected through the inputs. The two subsystems model are the following:

$$\frac{dx_1}{dt} = \begin{bmatrix} \frac{-1}{\tau_1} & \frac{A_3}{A_1\tau_3} \\ 0 & \frac{-1}{\tau_3} \end{bmatrix} x_1 + \begin{bmatrix} \frac{\gamma_a}{A_1} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \frac{(1-\gamma_b)}{A_3} \end{bmatrix} u_2.$$

$$\frac{dx_2}{dt} = \begin{bmatrix} \frac{-1}{\tau_2} & \frac{A_4}{A_2\tau_4} \\ 0 & \frac{-1}{\tau_4} \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ \frac{(1-\gamma_a)}{A_4} \end{bmatrix} u_1 + \begin{bmatrix} \frac{\gamma_b}{A_2} \\ 0 \end{bmatrix} u_2.$$

where $x_1 = (h_1, h_3)$, $x_2 = (h_2, h_4)$, $u_1 = q_a$ and $u_2 = q_b$.

The overall control objective is to control the level of tanks 1 and 2 while fulfilling the constraints on the levels and on the inputs.

5.6.2 Simulations

In the test, five references have been considered: $y_{t,1} = (0.3, 0.3)$, $y_{t,2} = (1.25, 1.25)$, $y_{t,3} = (0.35, 0.8)$, $y_{t,4} = (1, 0.8)$ and $y_{t,5} = (h_1^0, h_2^0)$. Notice that $y_{t,3}$ is not an equilibrium output for the system. The initial state is $x_0 = (0.65, 0.65, 0.6658, 0.6242)$. Notice also that the constraints on the model are coupled due to the dynamic. The setups for the two distributed controllers are the followings:

- Agent 1: $Q_1 = I_2$, $R_1 = 0.01I_1$, N=3, $w_1 = 0.5$.
- Agent 2: $Q_2 = I_2$, $R_2 = 0.01I_1$, N=3, $w_2 = 0.5$.

The number of iterations of the suboptimal optimization algorithm has been chosen as $\bar{p} = 1$. The gain K is chosen as the one of the LQR and the matrix P is the solution of the Riccati equation. The invariant set for tracking has been calculating for the gain matrix K. The offset cost function $V_O(y_s, y_t)$ has been chosen following chapter 2. The projection of the maximal invariant set for tracking onto y, Ω_Y, the projection of the region of attraction onto Y, $\mathcal{Y}_3$ and the set of equilibrium levels $\mathcal{Y}_s$, are plotted in figure 5.1. The results of the simulation are plotted in Figures 5.2 and 5.3.

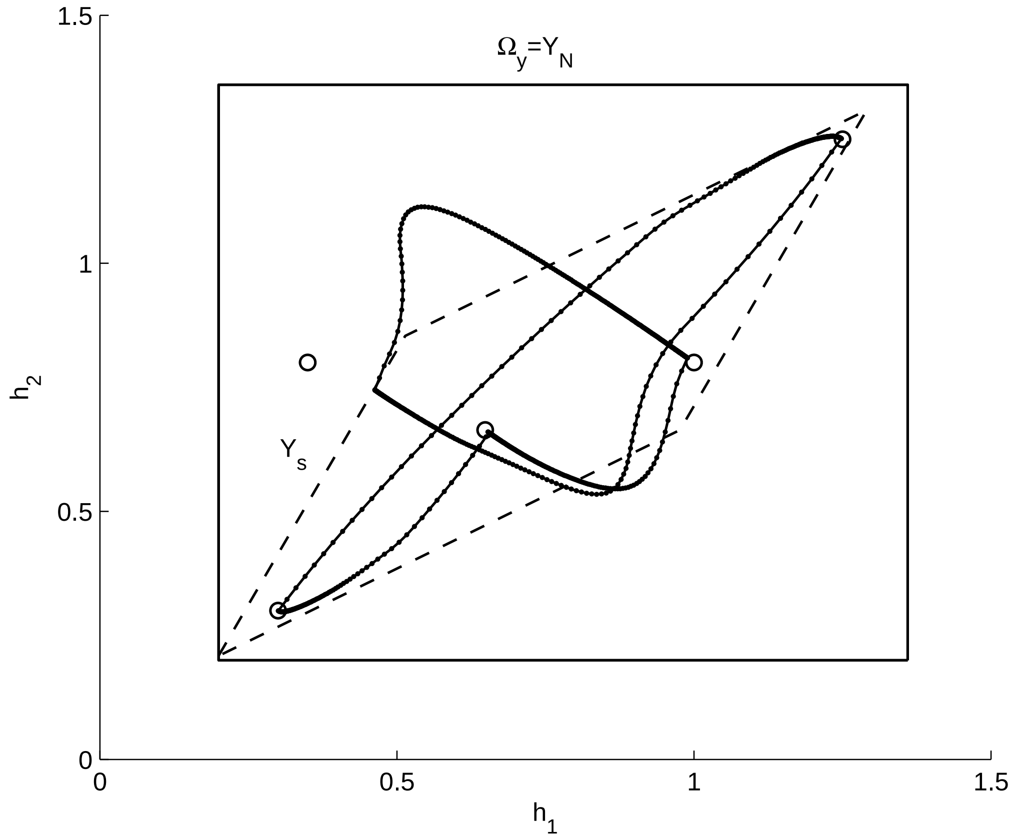


Figure 5.1: Steady output set, $\mathcal{Y}_s$, and projection of $\Omega_{t,K}$ onto Y, $\Omega_{t,Y}$.

In figure 5.2 the levels of tanks 1 and 2 are plotted. The evolutions of the systems are plotted in solid lines, while the references and the artificial references are plotted

respectively in dashed-dotted and dashed lines.
See how the controller always steers the system to the given reference, and how the evolutions follow the artificial references while the real one are unfeasible. The artificial references are the optimal steady states that the system can reach in that moment with that prediction horizons. Looking at their evolution, we can see the moment in which the desired reference becomes a feasible point for the optimization problem. This moment in the figure is the moment in which the artificial reference (dashed-dotted line) (practically) reach the real one (dashed line). Notice, also, that in the second change of reference, when the target setpoint in unreachable (due to the constraints), the controller steers the system to the optimal steady state of the centralized problem. The evolutions using the cooperative distributed MPC presented in (Rawlings and Mayne, 2009, Chapter 6, pp. 456-458), in case of a decentralized solution of the target problem, are plotted in dotted lines. See how this controller steers the system to a different steady state, given by the solution of a decentralized target problem.

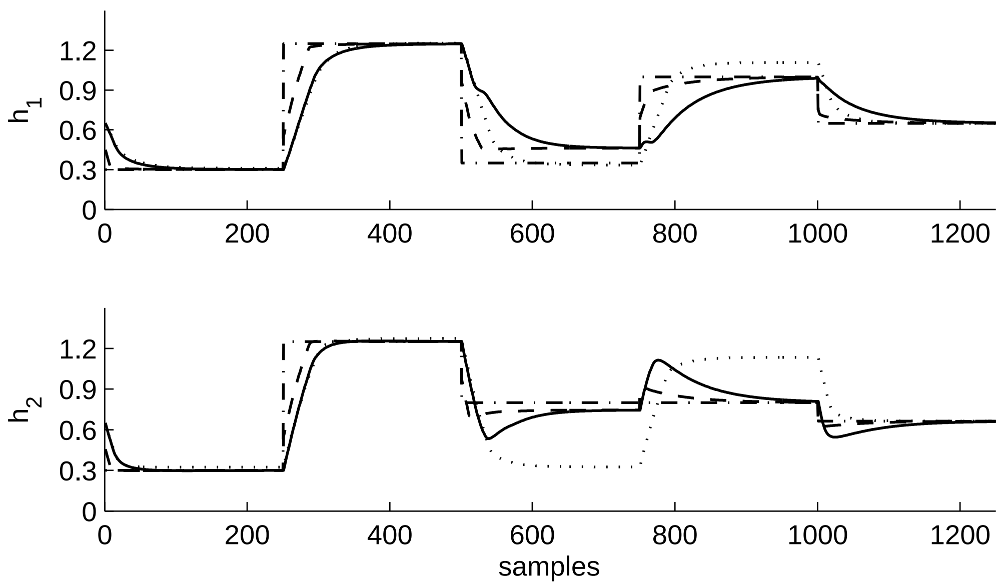


Figure 5.2: Time evolution of tanks 1 and 2 levels.

In Figure 5.3 the levels of tanks 3 and 4 and the control actions, which are the flows from the pumps, are plotted in solid lines. The artificial references are also plotted in dashed lines. state of the centralized problem. The evolutions using the cooperative distributed MPC presented in (Rawlings and Mayne, 2009, Chapter 6, pp. 456-458), in case of a decentralized solution of the target problem, are plotted in dotted lines.

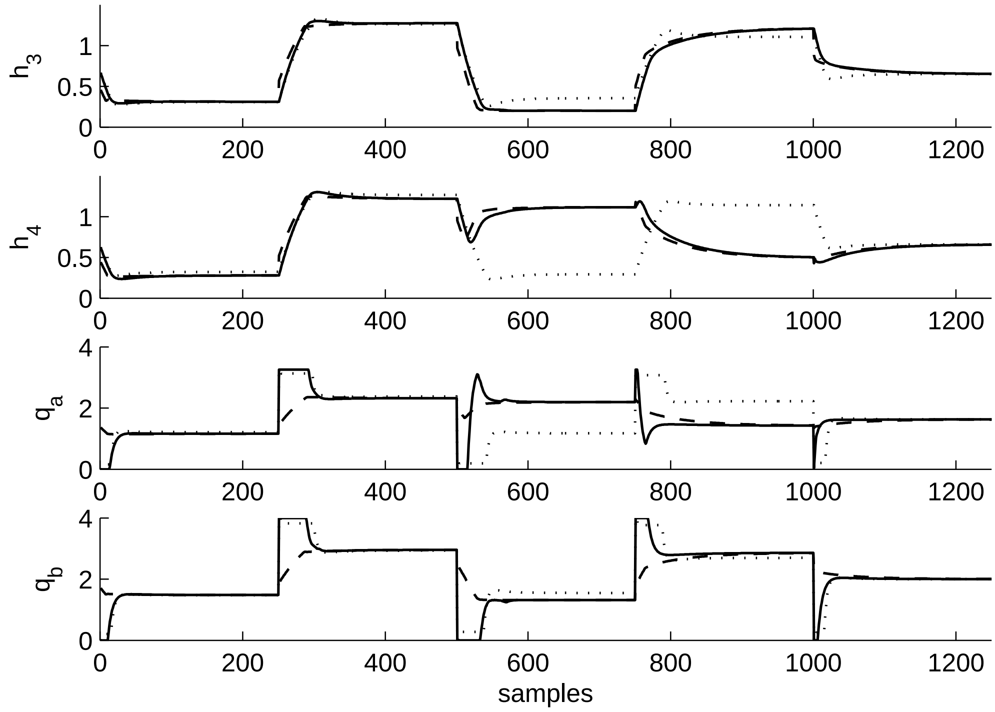


Figure 5.3: Time evolution of tanks 3 and 4 levels and of the flows.

The performance of the cooperative MPC for tracking with only one iteration has been compared to the same controller with 10 iterations of the optimization algorithm. In figures 5.4 and 5.5 a detail of this comparison is presented. This detail refers to the fourth change o reference. The cooperative MPC for tracking with one iteration is plotted in solid black line, while the one with 10 iterations is plotted in solid blue line. See how, increasing the number of iterations, the controller seem to be faster and the overshoots to be reduced. The changes in the control actions, also, result to be smoother.

The performance of the two controller have been also compared calculating the following closed-loop control performance measure:

$$\Phi \ = \ \sum_{k=0}^{T} \|x(k) - x_t\|_Q^2 + \|u(k) - u_t\|_R^2 - (\|x_s^* - x_t\|_Q^2 + \|u_s^* - u_t\|_R^2)$$

where T is the simulation time. The results in table 5.1 show that the closed-loop performance using 10 iterations are better than using only one iteration, at expanse of a heavier computational burden.

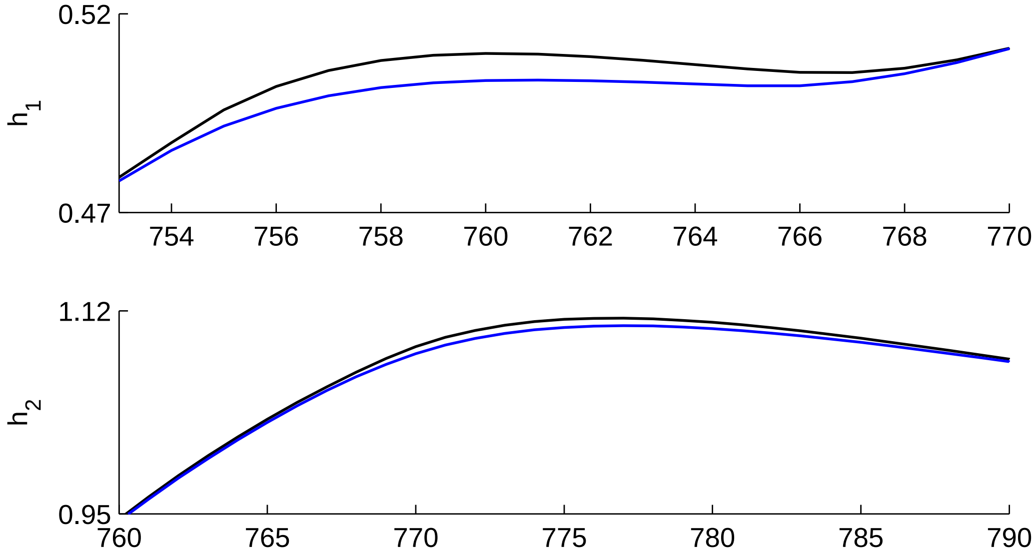


Figure 5.4: Time evolution of tanks 1 and 2 levels.

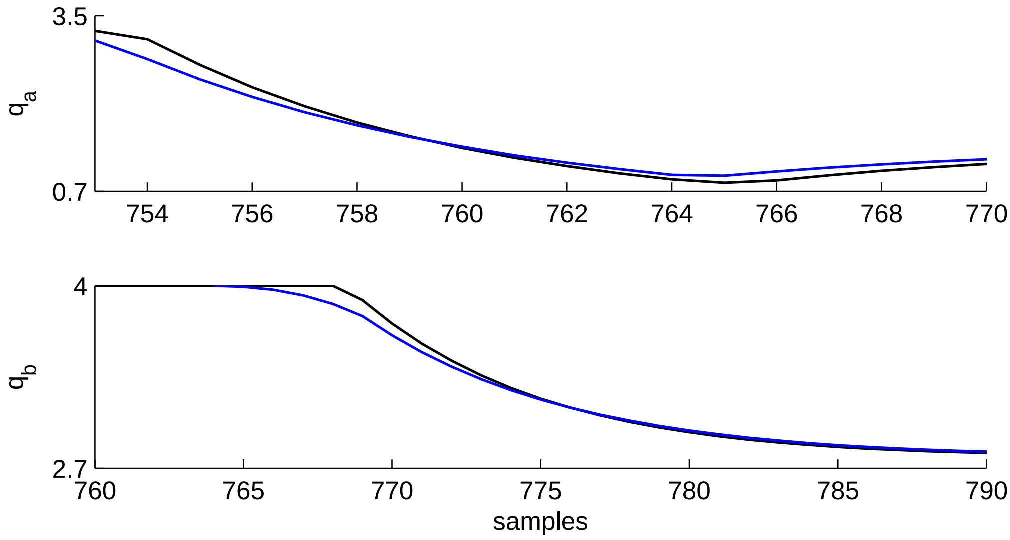


Figure 5.5: Time evolution of tanks 3 and 4 levels and of the flows.

5.7 Conclusion

In this chapter, a cooperative distributed linear model predictive control strategy has been proposed, applicable to any finite number of subsystems, for solving the problem

Table 5.1: Comparison of controller performance

	Φ	Average calculation time (s)
Coop. MPC for tracking (1 iteration)	173.7353	0.1150
Coop. MPC for tracking (10 iterations)	173.4893	0.6341

of tracking non-zero setpoints. The proposed controller is able to steer the system to any admissible setpoint in an admissible way. Feasibility under any changing of the target steady state and convergence to the centralized optimum are ensured. Under some assumptions, it is proved that the proposed controller steers the system to the target if this is admissible. If not, the controller converges to an admissible steady state optimum according to the offset cost function.

CHAPTER 6

MPC for tracking constrained nonlinear systems

6.1 Introduction

This chapter is dedicated to the case of nonlinear systems and to the design of an MPC control strategy for tracking a (possibly time varying) setpoint.

Tracking control of constrained nonlinear systems is an interesting problem due to the nonlinear nature of many processes in industry mainly when large transitions are required, as in the case of changing operating point.

In (Findeisen et al., 2000) a nonlinear predictive control for setpoint families is presented, which considers a pseudolinearization of the system and a parametrization of the setpoints. The stability is ensured thanks to a quasi-infinite nonlinear MPC strategy, but the solution of the tracking problem is not considered.

In (Magni et al., 2001b) an output feedback receding horizon control algorithm for nonlinear discrete-time systems is presented, which solves the problem of tracking exogenous signals and asymptotically rejecting disturbances generated by a properly defined exosystem. In (Magni and Scattolini, 2005) an MPC algorithm for nonlinear systems is proposed, which guarantees local stability and asymptotic tracking of constant references. This algorithm need the presence of an integrator preliminarily plugged in front of the system to guarantee the solution of the asymptotic tracking problem. In (Magni and Scattolini, 2007) an MPC algorithm for continuous-time, possibly non-square nonlinear systems is presented. The algorithm guarantees the tracking of asymptotically constant reference signals by means of a control scheme were the integral action is directly imposed on the error variables rather than on the control moves.

Another approach to the tracking of nonlinear systems problem are the so-called reference governors (Angeli and Mosca, 1999; Bemporad et al., 1997; Gilbert and Kolmanovsky, 2002). A reference governor is a nonlinear device which manipulates on-line a command input to a suitable pre-compensated system so as to satisfy constraints. This can be seen as adding an artificial reference, computed at each sampling time to ensure the admissible evolution of the system, converging to the desired reference.

In (Chisci and Zappa, 2003) the tracking problem for constrained linear systems is solved by means of an approach called dual mode: the dual mode controller operates as a regulator in a neighborhood of the desired equilibrium wherein constraints are feasible, while it switches to a feasibility recovery mode, whenever this is lost due to a setpoint change, which steers the system to the feasibility region of the MPC as quickly as possible. In (Chisci et al., 2005) this approach is extended to nonlinear systems, considering constraint-admissible invariant sets as terminal regions, obtained by means of a LPV model representation of the nonlinear plant.

In (Limon et al., 2008a; Ferramosca et al., 2009a) an MPC for tracking of constrained linear systems is proposed, which is able to lead the system to any admissible setpoint in an admissible way. The main characteristics of this controller are: an artificial steady state is considered as a decision variable, a cost that penalizes the error with the artificial steady state is minimized, an additional term that penalizes the deviation between the artificial steady state and the target steady state is added to the cost function (the so-called *offset cost function*) and an invariant set for tracking is considered as extended terminal constraint. This controller ensures that under any change of the target steady state, the closed-loop system maintains the feasibility of the controller and ensures the convergence to the target if admissible.

In this chapter, this controller is extended to the case of nonlinear constrained systems. Three formulations of the controller are presented, which consider respectively the cases of terminal equality constraint, terminal inequality constraint and no terminal constraint.

6.2 Problem statement

Consider a system described by a nonlinear invariant discrete time model

$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, $y \in \mathbb{R}^p$ is the controlled output and x^+ is the successor state. The function model $f(x,u)$ is assumed to be continuous at any equilibrium point. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u})$ where $x = \phi(0; x, \mathbf{u})$. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively. The system is subject to hard constraints on state and control:

$$x(k) \in X, \quad u(k) \in U \tag{6.2}$$

for all $k \geq 0$, where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are closed sets.

The steady state, input and output of the plant (x_s, u_s, y_s) are such that (6.1) is fulfilled, i.e.

$$x_s = f(x_s, u_s) \tag{6.3}$$

$$y_s = h(x_s, u_s) \tag{6.4}$$

Let define the set of admissible equilibrium states as

$$\mathcal{Z}_s = \{(x,u) \in X \times U : x = f(x,u)\} \tag{6.5}$$

$$\mathcal{X}_s = \{x \in X : \exists u \in U \text{ s. t. } (x,u) \in \mathcal{Z}_s\} \tag{6.6}$$

$$\mathcal{Y}_s = \{y = h(x,u) : (x,u) \in \lambda \mathcal{Z}_s\} \tag{6.7}$$

where $\lambda \in (0,1)$.

Assumption 6.1 *Assuming that the system is observable, the output of the system univocally defines each triplet (x_s, u_s, y_s), i.e.*

$$x_s = g_x(y_s), \; u_s = g_u(y_s) \tag{6.8}$$

Remark 6.2 *If system 6.1 is not observable, it is convenient to find a parameter $\theta \in \mathbb{R}^{n_\theta}$ such such that the triplet (x_s, u_s, y_s) is univocally defined.*

The problem we consider is the design of an MPC controller $\kappa(x, y_t)$ to track a (possible time-varying) target steady output y_t, such that the system is steered as close as possible to the target while fulfilling the constraints.

6.3 MPC for tracking

In this section, the proposed MPC for tracking is presented. The aim of this novel formulation is to guarantee recursive feasibility for any (possibly changing) output target to be tracked and, if possible, the convergence of the output of the plant to the target.

The cost function of the proposed MPC is given by:

$$V_N(x, y_t; \mathbf{u}, y_s) = \sum_{j=0}^{N-1} \ell((x(j)-x_s), (u(j)-u_s)) + V_O(y_s - y_t)$$

where $x(j) = \phi(j; x, \mathbf{u})$, $x_s = g_x(y_s)$, $u_s = g_u(y_s)$ and y_t is the target of the controlled variables.

The controller is derived from the solution of the optimization problem $P_N(x, y_t)$ given by:

$$
\begin{aligned}
&\min_{\mathbf{u}, y_s} V_N(x, y_t; \mathbf{u}, y_s) \\
&\quad s.t. \\
&\qquad x(0) = x, \\
&\qquad x(j+1) = f(x(j), u(j)), && j = 0, \cdots, N-1 \\
&\qquad x(j) \in X,\ u(j) \in U && j = 0, \cdots, N-1 \\
&\qquad x_s = g_x(y_s) \\
&\qquad u_s = g_u(y_s) \\
&\qquad y_s \in \mathcal{Y}_s \\
&\qquad x(N) = x_s
\end{aligned}
$$

The optimal cost and the optimal decision variables will be denoted as $V_N^0(x, y_t)$ and $(\mathbf{u}^0, y_s^0)$ respectively. Considering the receding horizon policy, the control law is given by

$$\kappa_N(x, y_t) = u^0(0; x, y_t)$$

Since the set of constraints of $P_N(x, y_t)$ does not depend on y_t, its feasibility region does not depend on the target operating point y_t. Then there exists a region $\mathcal{X}_N \subseteq X$ such that for all $x \in \mathcal{X}_N$ and for all $y_t \in \mathbb{R}^p$, $P_N(x, y_t)$ is feasible. This is the set of states that can reach any admissible equilibrium point in N steps.

Consider the following assumption on the controller parameters:

Assumption 6.3

1. *The model function $f(x, u)$ is continuous in $\mathcal{Z}_s$.*
2. *There exists a $\mathcal{K}$ function α_ℓ such that the stage cost function fulfills $\ell(z, v) \geq \alpha_\ell(|z|)$*
3. *Let the offset cost function $V_O : \mathbb{R}^p \to \mathbb{R}$ be a convex positive definite function such that the minimizer*

$$y_s^* = arg \min_{y_s \in \mathcal{Y}_s} V_O(y_s - y_t)$$

is unique. Moreover, there exists a $\mathcal{K}$ function α_O such that $V_O(y_s - y_t) \geq \alpha_O(|y_s - y_s^*|)$.

4. *The system is weakly controllable at any equilibrium point $(x_s, u_s) \in \mathcal{Z}_s$ (Rawlings and Mayne, 2009). That is, for any $(\mathbf{u}, y_s)$ feasible solution of $P_N(x, y_t)$, there exists a $\mathcal{K}_\infty$ function γ such that,*

$$\sum_{i=0}^{N-1} |u(i) - u_s| \leq \gamma(|x - x_s|)$$

holds for all $x \in \mathcal{X}_N$.

5. *The set of admissible output $\mathcal{Y}_s$ is a convex set.*

Remark 6.4 *If $\mathcal{Y}_s = \{y = h(x, u) : (x, u) \in \lambda\mathcal{Z}_s\}$ is not convex, then set $\mathcal{Y}_s$ must be chosen as a convex set contained in $\{y = h(x, u) : (x, u) \in \lambda\mathcal{Z}_s\}$.*

The following theorem proves asymptotic stability and constraints satisfaction of the controlled system.

Theorem 6.5 (Stability) *Consider that assumptions 6.1 and 6.3 hold and consider a given target operation point y_t. Then for any feasible initial state $x_0 \in \mathcal{X}_N$, the system controlled by the proposed MPC controller $\kappa_N(x, y_t)$ is stable, converges to an equilibrium point, fulfils the constraints along the time and besides*

(i) If $y_t \in \mathcal{Y}_s$ then $\lim_{k\to\infty} |y(k) - y_t| = 0$.

(ii) If $y_t \notin \mathcal{Y}_s$, then $\lim_{k\to\infty} |y(k) - y_s^*| = 0$, where

$$y_s^* = \arg\min_{y_s \in \mathcal{Y}_s} V_O(y_s - y_t)$$

Proof: Consider that $x \in \mathcal{X}_N$ at time k, then the optimal cost function is given by $V_N^0(x, y_t) = V_N(x, y_t; \mathbf{u}^0(x), y_s^0(x))$, where $(\mathbf{u}^0(x), y_s^0(x))$ defines the optimal solution of $P_N(x, y_t)$ and $\mathbf{u}^0(x) = \{u^0(0; x), u^0(1; x), ..., u^0(N-1; x)\}$. Notice that $u^0(0; x) = \kappa_N(x, y_t)$. The resultant optimal state sequence associated to $\mathbf{u}^0(x)$ is given by $\mathbf{x}^0(x) = \{x^0(0; x), x^0(1; x), ..., x^0(N-1; x), x^0(N; x)\}$, where $x^0(0; x) = x$, $x^0(1; x) = x^+$ and $x^0(N; x) = x_s^0(x) = g_x(y_s^0(x))$.

As standard in MPC (Mayne et al., 2000; Rawlings and Mayne, 2009, Chapter 2), define the successor state at time $k+1$, $x^+ = f(x, \kappa_N(x, y_t))$ and define also the following sequences:

$$\begin{aligned} \tilde{\mathbf{u}} &\triangleq [u^0(1; x), \cdots, u^0(N-1; x), u_s^0(x)] \\ \tilde{y}_s &\triangleq y_s^0(x) \end{aligned}$$

where $u_s^0(x) = g_u(y_s^0(x))$. It is easy to derive that $(\tilde{\mathbf{u}}, \tilde{y}_s)$ is a feasible solution for the optimization problem $P_N(x^+)$. Therefore, $\mathcal{X}_N$ is an admissible positive invariant set for the closed-loop system and hence the control law is well-defined and the constraints are fulfilled throughout the system evolution.

The state sequence due to $(\tilde{\mathbf{u}}, \tilde{y}_s)$ is $\tilde{\mathbf{x}} = \{x^0(1; x), x^0(2; x), ..., x^0(N; x), x^0(N+1; x)\}$, where $x^0(N; x) = x_s^0(x)$ and $x^0(N+1; x) = f(x^0(N; x), u_s^0(x)) = x_s^0(x)$. Hence,

$$\tilde{\mathbf{x}} = \{x^0(1; x), x^0(2; x), ..., x_s^0(x), x_s^0(x)\}$$

which is clearly feasible. Compare now the optimal cost $V_N^0(x, y_t)$, with the cost given by $(\tilde{\mathbf{u}}, \tilde{y}_s)$, $\tilde{V}_N(x^+, y_t; \tilde{\mathbf{u}}, \tilde{y}_s)$. Taking into account the properties of the feasible nominal trajectories for x^+, Assumption 6.3 and using standard procedures in MPC (Mayne et al., 2000; Rawlings and Mayne, 2009, Chapter 2) it is possible to obtain:

$$\begin{aligned}\tilde{V}_N(x^+, y_t; \tilde{\mathbf{u}}, \tilde{y}_s) - V_N^0(x, y_t) &= -\ell((x - x_s^0(x)), (u^0(0; x) - u_s^0(x))) - V_O(y_s^0 - y_t) \\ &\quad + \ell((x(N; x) - x_s^0(x)), (u_s^0(x) - u_s^0(x))) + V_O(y_s^0 - y_t) \\ &= -\ell((x - x_s^0(x)), (u^0(0; x) - u_s^0(x)))\end{aligned}$$

By optimality, we have that $V_N^0(x^+, y_t) \leq \tilde{V}_N(x^+, y_t; \tilde{\mathbf{u}}, \tilde{y}_s)$ and then:

$$\begin{aligned}V_N^0(x^+, y_t) - V_N^0(x, y_t) &\leq -\ell((x - x_s^0(x)), (u^0(0; x) - u_s^0(x))) \\ &= -\ell((x - x_s^0(x)), (\kappa_N(x, y_t) - u_s^0(x)))\end{aligned}$$

Taking into account that the cost function is a positive definite function, we have that:

$$\lim_{k\to\infty} |x(k) - x_s^0(x(k))| = 0, \quad \lim_{k\to\infty} |u(k) - u_s^0(x(k))| = 0$$

Now, it is proved that the system converges to an equilibrium point. Pick an $\varepsilon > 0$, then there exists a $k(\varepsilon)$ such that for all $k \geq k(\varepsilon)$, $|x(k) - x_s^0(x(k))| < \varepsilon$ and $|u(k) - u_s^0(x(k))| < \varepsilon$. Moreover, $f(x, u)$ is continuous in $\mathcal{Z}_s$, and hence there exists $\alpha(\varepsilon)$ such that $|f(x(k), u(k)) - f(x_s^0(x(k)), u_s^0(x(k)))| \leq \alpha(\varepsilon)$. Then, removing the time dependence for the sake of simplicity, it is inferred that

$$\begin{aligned}|x^+ - x| &= |x^+ - x_s^0(x) + x_s^0(x) - x| \\ &\leq |x^+ - x_s^0(x)| + |x_s^0(x) - x| \\ &= |f(x, u) - f(x_s^0(x), u_s^0(x))| + |x_s^0(x) - x| \\ &\leq \alpha(\varepsilon) + \varepsilon\end{aligned}$$

Therefore, for a given $\varepsilon > 0$, there exists a $k(\varepsilon)$ such that $|x^+ - x| \leq \alpha(\varepsilon) + \varepsilon$. Hence, the system converges to a steady state x_∞ and this is such that $x_\infty = x_s^0(x_\infty) \in \mathcal{X}_s$.

Using lemma 6.18 (see the Appendix section of this chapter), it is proved that (x_∞, u_∞) is the optimal steady state of the system, that is $(x_\infty, u_\infty) = (x_s^*, u_s^*)$, where $x_s^* = g_x(y_s^*)$ and $u_s^* = g_u(y_s^*)$.

Finally, the fact that (x_s^*, u_s^*) is a stable equilibrium point for the closed-loop system is proved. That is, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $|x(0) - x_s^*| \leq \delta$, then $|x(k) - x_s^*| \leq \varepsilon$.

To this aim, define the function $W(x, y_t) = V_N^0(x, y_t) - V_O(y_s^* - y_t)$. Then, $W(x_s^*, y_t) = 0$. This function is such that $\alpha_W(|x - x_s^*|) \leq W(x, y_t) \leq \beta_W(|x - x_s^*|)$, where α_W and β_W are suitable $\mathcal{K}_\infty$ functions. In fact:

- $W(x, y_t) \geq \alpha_l(|x - x_s^0|) + \alpha_O(|x_s^0 - x_s^*|)$. This comes from the fact that the stage cost function is a positive definite function and from the definition of V_O (assumption 6.3). Hence

$$\begin{aligned} W(x, y_t) &\geq \alpha_W(|x - x_s^0| + |x_s^0 - x_s^*|) \\ &\geq \alpha_W(|x - x_s^*|) \end{aligned}$$

- Notice that $V_N^0(x, y_t) \leq V_N(x, y_s^*) + V_O(y_s^* - y_t)$. Due to the weak controllability of x_s^* (assumption 6.3), there exists a $\mathcal{K}_\infty$ function β_W such that $V_N(x, y_s^*) \leq \beta_W(|x - x_s^*|)$. Hence $W(x, y_t) \leq \beta_W(|x - x_s^*|)$.

Then, $\alpha_W(|x(k) - x_s^*|) \leq W(x(k), y_t) \leq W(x(0), y_t) \leq \beta_W(|x - x_s^*|)$ and, hence, $|x(k) - x_s^*| \leq \alpha_W^{-1} \circ \beta_W(|x(0) - x_s^*|)$. So, picking $\delta = \beta_W^{-1} \circ \alpha_W(\varepsilon)$, then $|x(k) - x_s^*| \leq \alpha_W^{-1} \circ \beta_W(\delta) \leq \varepsilon$, proving the stability of x_s^*.

Then, for all initial state $x_0 \in \mathcal{X}_N$, the closed-loop system converges to an asymptotic stable equilibrium point (x_s^*, u_s^*) and its domain of attraction is $\mathcal{X}_N$. ■

6.4 Properties of the proposed controller

Besides the asymptotic stability property, this controller also provides the following properties.

6.4.1 Changing operation points

Considering that problem $P_N(x, y_t)$ is feasible for any y_t, then the proposed controller is able to track changing operation points maintaining the recursive feasibility and admissibility.

6.4.2 Stability for any admissible steady state

Since property 6.4.1 holds for any value of the horizons N, it can be derived that the proposed controller is able to track any admissible setpoint $y_t \in \mathcal{Y}_s$, even for $N = 1$, if the system starts from a feasible initial state.
Typically, the starting point of the controller is an equilibrium point. If this point is reachable, i.e. $x_0 = g_x(y_0)$, $y_0 \in \mathcal{Y}_s$, then the system can be steered to any reachable equilibrium point, for any N.

6.4.3 Enlargement of the domain of attraction

The domain of attraction of the MPC is the set of states that can be admissible steered to x_s^*. The fact that for the proposed controller this set is considered with respect to any equilibrium point, makes this set (potentially) larger than the one calculated for regulation to a fixed equilibrium point. Consequently, the domain of attraction of the proposed controller is (potentially) larger than the domain of the standard MPC. This property is particularly interesting for small values of the control horizon.

6.4.4 Steady state optimization

It is not unusual that the output target y_t is not contained in $\mathcal{Y}_s$. This may happen when there not exists an admissible operating point which steady output equals to the target or when the target is not a possible steady output of the system. To deal with this situation in predictive controllers, the standard solution is to add an upper level steady state optimizer to decide the best reachable target of the controller (Rao and Rawlings, 1999).
From the latter theorem it can be clearly seen that in this case, the proposed controller steers the system to the optimal operating point according to the offset cost function $V_O(\cdot)$. Then it can be considered that the proposed controller has a steady state optimizer built in and $V_O(\cdot)$ defines the function to optimize.

6.5 Local optimality

The proposed controller can be considered as a suboptimal controller due to the formulation of the stage cost. However, under mild assumption on the offset cost function, it is possible to prove that the MPC for tracking ensures the property of the local optimality. This property states that, in a neighborood of the terminal region, the constrained finite horizon MPC equals the infinite horizon one (Magni et al., 2001a; Hu and Linnemann, 2002). The standard MPC control law for regulation to a target y_t, $k_N^r(x,y_t)$, derived from the following optimization problem $P_N^r(x,y_t)$:

$$
\begin{aligned}
V_N^{r,0}(x,y_t) \quad &= \quad \min_{\mathbf{u},y_s} \sum_{j=0}^{N-1} \ell((x(j)-x_s),(u(j)-u_s)) && (6.9)\\
s.t. \quad & && (6.10)\\
& x(0) = x, && (6.11)\\
& x(j+1) = f(x(j),u(j)) \quad j=0,\cdots,N-1 && (6.12)\\
& x(j) \in X,\ u(j) \in U \quad j=0,\cdots,N-1 && (6.13)\\
& x_s = g_x(y_s) && (6.14)\\
& u_s = g_u(y_s) && (6.15)\\
& y_s \in \mathcal{Y}_s && (6.16)\\
& x(N) = x_s && (6.17)\\
& |y_s - y_t|_q = 0 && (6.18)
\end{aligned}
$$

ensures the local optimality. The domain of attraction of this problem is noted as $\mathcal{X}_N^r(y_t)$ As in the linear case (see chapter 2), in the MPC for tracking, this property can be ensured by means of a suitable choice of the offset cost function.

Assumption 6.6 *Let the offset cost function fulfill assumption 6.3. Moreover there exists a positive constant α such that:*

$$V_O(y_s - y_t) \geq \alpha|y_s - y_t|$$

Then, we can state the following property:

Property 6.7 *Consider that assumptions 6.1, 6.3 and 6.6 hold. Then there exists an α^* such that for all $\alpha \geq \alpha^*$ and for all $x \in \mathcal{X}_N^r(y_t)$, the MPC for tracking equals the MPC for regulation, that is $k_N(x,y_t) = k_N^r(x,y_t)$*

Proof: Define problem $P^r_{N,\alpha}(x, yt)$ as:

$$
\begin{aligned}
V^{r,0}_{N,\alpha}(x, y_t) \quad = \quad & \min_{\mathbf{u},y_s} \sum_{j=0}^{N-1} \ell((x(j)-x_s),(u(j)-u_s)) + \alpha|y_s - y_t|_p \\
s.t. \quad & \\
& x(0) = x, \\
& x(j+1) = f(x(j), u(j)) \quad j=0,\cdots,N-1 \\
& x(j) \in X,\ u(j) \in U \quad j=0,\cdots,N-1 \\
& x_s = g_x(y_s) \\
& u_s = g_u(y_s) \\
& y_s \in \mathcal{Y}_s \\
& x(N) = x_s
\end{aligned}
$$

where $|.|_p$ is the dual of norm $|.|_q$[1]. Then, problem $P^r_{N,\alpha}(x, yt)$ results from problem $P^r_N(x, yt)$ with the last constraint posed as an exact penalty function (Luenberger, 1984). Therefore, there exists a finite constant $\alpha^* > 0$ such that for all $\alpha \geq \alpha^*$, $V^{r,0}_{N,\alpha}(x, y_t) = V^{r,0}_N(x, y_t)$ for all $x \in \mathcal{X}^r_N(y_t)$ (Luenberger, 1984; Boyd and Vandenberghe, 2006).

Consider now, problem $P_N(x, y_t)$. Taking $V_O(y_s - y_t) = \alpha|y_s - y_t|_p$, with $\alpha \geq \alpha^*$, we can state that $V^0_N(x, y_t) = V^{r,0}_N(x, y_t)$ for all $x \in \mathcal{X}^r_N(y_t)$. ■

6.6 MPC for tracking with terminal inequality constraint

This formulation of the proposed controller, as standard in MPC, relies on the calculation of a suit terminal control law $u = \kappa(x, y_s)$ and on the use of an invariant set as terminal constraint. The knowledge of this control law allows to use a prediction horizon N_p larger than the control horizon N_c, in such a way that the control action are extended using the terminal control law (Magni et al., 2001a). The proposed cost

[1]The dual $|.|_p$ of a given norm $|.|_q$ is defined as $|u|_p \triangleq \max_{|v|_q \leq 1} u'v$. For instance, $p = 1$ if $q = \infty$ and vice versa, or $p = 2$ if $q = 2$ (Luenberger, 1984).

function of the MPC is given by:

$$\begin{aligned}V_{N_c,N_p}(x,y_t;\mathbf{u},y_s) = &\sum_{j=0}^{N_c-1}\ell((x(j)-x_s),(u(j)-u_s)) + \sum_{j=N_c}^{N_p-1}\ell((x(j)-x_s),(\kappa(x(j),y_s)-u_s))\\ &+V_f(x(N_p)-x_s,y_s)+V_O(y_s-y_t)\end{aligned}$$

where $x(j) = \phi(j;x,\mathbf{u})$, $x_s = g_x(y_s)$, $u_s = g_u(y_s)$ and $y_s = g_y(y_s)$; y_t is the target of the controlled variables.

The controller is derived from the solution of the optimization problem $P_{N_c,N_p}(x,y_t)$ given by:

$$\begin{aligned}
&\min_{\mathbf{u},y_s} V_{N_c,N_p}(x,y_t;\mathbf{u},y_s)\\
&\quad s.t.\\
&\qquad x(0) = x,\\
&\qquad x(j+1) = f(x(j),u(j)), && j=0,\cdots,N_c-1\\
&\qquad x(j)\in X,\ u(j)\in U, && j=0,\cdots,N_c-1\\
&\qquad x(j+1) = f(x(j),\kappa(x(j),y_s)), && j=N_c,\cdots,N_p-1\\
&\qquad x(j)\in X,\ \kappa(x(j),y_s)\in U && j=N_c,\cdots,N_p-1\\
&\qquad x_s = g_x(y_s),\ u_s = g_u(y_s)\\
&\qquad (x(N_p),y_s)\in\Gamma
\end{aligned}$$

The optimal cost and the optimal decision variables will be denoted as $V^0_{N_c,N_p}(x,y_t)$ and $(\mathbf{u}^0,y_s^0)$ respectively. Considering the receding horizon policy, the control law is given by

$$\kappa_{N_c,N_p}(x,y_t) = u^0(0;x,y_t)$$

Since the set of constraints of $P_{N_c,N_p}(x,y_t)$ does not depend on y_t, its feasibility region does not depend on the target operating point y_t. Then there exists a region $\mathcal{X}_{N_c,N_p}\subseteq X$ such that for all $x\in\mathcal{X}_{N_c,N_p}$ and for all $y_t\in\mathbb{R}^p$, $P_{N_c,N_p}(x,y_t)$ is feasible.

In order to derive the stability conditions, it is convenient to extend the notion of invariant set for tracking introduced in (Limon et al., 2008a) to the nonlinear case.

Definition 6.8 (Invariant set for tracking) *A set $\Gamma\subset\mathbb{R}^n\times\mathbb{R}^p$ is an (admissible) invariant set for tracking for system 6.1 controlled by $\kappa(x,y_s)$ if for all $(x,y_s)\in\Gamma$ we have that $x\in X$, $y_s\in\mathcal{Y}_s$, $\kappa(x,y_s)\in U$, and $(f(x,\kappa(x,y_s)),y_s)\in\Gamma$.*

This set can be read as the set of initial states and setpoints that provides an admissible evolution of the system 6.1 controlled by $u = \kappa(x, y_s)$.

The following conditions on the terms of the proposed controller are assumed:

Assumption 6.9

1. *Let the function $g_x(y_s)$ be Lipshitz continuous in $\mathcal{Y}_s$.*
2. *Let $k(x, y_s)$ be a control law such that for all $y_s \in \mathcal{Y}_s$, the equilibrium point $x_s = g_x(y_s)$ and $u_s = g_u(y_s)$ is an asymptotically stable equilibrium point for the system $x^+ = f(x, k(x, y_s))$.*
3. *Let Γ be an invariant set for tracking for the system $x^+ = f(x, k(x, y_s))$.*
4. *Let $V_f(x - x_s, y_s)$ be a Lyapunov function for system $x^+ = f(x, k(x, y_s))$ such that for all $(x, y_s) \in \Gamma$*

$$\alpha_f(|x - x_s|) \leq V(x - x_s, y_s) \leq \beta_f(|x - x_s|)$$

and

$$V_f(f(x, k(x, y_s)) - x_s, y_s) - V_f(x - x_s, y_s) \leq -l(x - x_s, k(x, y_s) - u_s)$$

where $x_s = g_x(y_s)$ and $u_s = g_u(y_s)$, and where α_f and β_f are $\mathcal{K}$ functions. Moreover, there exist $b > 0$ and $\sigma > 1$ which verify $V_f(x_1 - x_2, y_s) \leq b\|x_1 - x_2\|^\sigma$ for all (x_1, y_s) and (x_2, y_s) contained in Γ.

Notice that the assumptions on the terminal ingredients are similar to the standard ones but extended to a set of equilibrium points.

The following theorem proves asymptotic stability and constraints satisfaction of the controlled system.

Theorem 6.10 (Stability) *Consider that assumptions 6.3 and 6.9 hold and consider a given target operation point y_t. Then for any feasible initial state $x_0 \in \mathcal{X}_{N_c,N_p}$, the system controlled by the proposed MPC controller $\kappa_{N_c,N_p}(x, y_t)$ is stable, converges to an equilibrium point, fulfils the constraints along the time and besides*

(i) If $y_t \in \mathcal{Y}_s$ then $\lim\limits_{k\to\infty} \|y(k) - y_t\| = 0$.

(ii) If $y_t \notin \mathcal{Y}_s$, then $\lim_{k\to\infty} \|y(k) - y_s^*\| = 0$, where

$$y_s^* = \arg\min_{y_s \in \mathcal{Y}_s} V_O(y_s - y_t)$$

Proof: Consider that $x \in \mathcal{X}_{N_C,N_P}$ at time k, then the optimal cost function is given by $V^0_{N_C,N_P}(x,y_t) = V_{N_C,N_P}(x,y_t;\mathbf{u}^0(x),y_s^0(x))$, where $(\mathbf{u}^0(x),y_s^0(x))$ defines the optimal solution of $P_{N_C,N_P}(x,y_t)$ and $\mathbf{u}^0(x) = \{u^0(0;x),u^0(1;x),...,u^0(N_c-1;x)\}$. Notice that $u^0(0;x) = \kappa_{N_c,N_p}(x,y_t)$. The resultant optimal state sequence associated to $\mathbf{u}^0(x)$ is given by $\mathbf{x}^0(x) = \{x^0(0;x),x^0(1;x),...,x^0(N_c-1;x),x^0(N_c;x),...,x^0(N_p;x)\}$, where $x^0(0;x) = x$, $x^0(1;x) = x^+$, and $x^0(N_c;x) = f(x^0(N_c-1;x),\kappa(x,y_s^0))$ and $x^0(N_p;x)$ is such that $(x^0(N_p;x),y_s^0(x)) \in \Gamma$.

As standard in MPC (Mayne et al., 2000; Rawlings and Mayne, 2009, Chapter 2), define the successor state at time $k+1$, $x^+ = f(x,\kappa_{N_c,N_p}(x,y_t))$ and define also the following sequences:

$$\begin{aligned}
\tilde{\mathbf{u}} &\triangleq [u^0(1;x),\cdots,u^0(N_c-1;x),\kappa(x,y_s^0)] \\
\tilde{y}_s &\triangleq y_s^0(x)
\end{aligned}$$

It is easy to derive that $(\tilde{\mathbf{u}},\tilde{y}_s)$ is a feasible solution for the optimization problem $P_{N_C,N_P}(x^+)$. Therefore, $\mathcal{X}_{N_C,N_P}$ is an admissible positive invariant set for the closed-loop system and hence the control law is well-defined and the constraints are fulfilled throughout the system evolution.

The state sequence due to $(\tilde{\mathbf{u}},\tilde{y}_s)$ is

$$\tilde{\mathbf{x}} = \{x^0(1;x),x^0(2;x),...,x^0(N_c;x),...,x^0(N_p;x),x^0(N_p+1;x)\}$$

where $x^0(N_p+1;x) = f(x^0(N_p;x),\kappa(x,y_s^0))$, which is clearly feasible. Compare now the optimal cost $V_N^0(x,y_t)$, with the cost given by $(\tilde{\mathbf{u}},\tilde{y}_s)$, $\tilde{V}_{N_C,N_P}(x^+,y_t;\tilde{\mathbf{u}},\tilde{y}_s)$. Taking into account the properties of the feasible nominal trajectories for x^+, Assumption 6.9 and using standard procedures in MPC (Mayne et al., 2000; Rawlings and Mayne, 2009, Chapter 2) it is possible to obtain:

$$\begin{aligned}
\tilde{V}_{N_C,N_P}(x^+,y_t;\tilde{\mathbf{u}},\tilde{y}_s) - V^0_{N_C,N_P}(x,y_t) = & -\ell((x-x_s^0(x)),(u^0(0;x)-u_s^0(x))) \\
& -V_f(x^0(N_p;x)-x_s^0(x)) - V_O(y_s^0-y_t) \\
& +\ell((x(N_p;x)-x_s^0(x)),(\kappa(x,y_s^0)-u_s^0(x))) \\
& +V_f(f(x^0(N_p;x),\kappa(x,y_s^0)) - x_s^0(x)) + V_O(y_s^0-y_t)
\end{aligned}$$

Since by optimality, $V^0_{N_C,N_P}(x^+,y_t) \leq \tilde{V}_{N_C,N_P}(x^+,y_t;\tilde{\mathbf{u}},\tilde{y}_s)$ and given the definition of $V_f(x-x_s)$ from assumption 6.9, then:

$$\begin{aligned} V^0_{N_C,N_P}(x^+,y_t) - V^0_{N_C,N_P}(x,y_t) &\leq -\ell((x-x^0_s(x)),(u^0(0;x)-u^0_s(x))) \\ &= -\ell((x-x^0_s(x)),(\kappa_{N_c,N_p}(x,y_t)-u^0_s(x))) \end{aligned}$$

Taking into account that the cost function is a positive definite function, we have that:

$$\lim_{k\to\infty}|x(k)-x^0_s(x(k))|=0, \quad \lim_{k\to\infty}|u(k)-u^0_s(x(k))|=0$$

Hence the system converges to an operating point (x^0_s,u^0_s), such that $x^0_s=g_x(y^0_s)$ and $u^0_s=g_u(y^0_s)$.

Now, it is proved that the system converges to an equilibrium point. Pick an $\varepsilon>0$, then there exists a $k(\varepsilon)$ such that for all $k\geq k(\varepsilon)$, $|x(k)-x^0_s(x(k))|<\varepsilon$ and $|u(k)-u^0_s(x(k))|<\varepsilon$. Moreover, $f(x,u)$ is continuous in $\mathcal{Z}_s$, and hence there exists $\alpha(\varepsilon)$ such that $|f(x(k),u(k))-f(x^0_s(x(k)),u^0_s(x(k)))|\leq\alpha(\varepsilon)$. Then, removing the time dependence for the sake of simplicity, it is inferred that

$$\begin{aligned} |x^+-x| &= |x^+-x^0_s(x)+x^0_s(x)-x| \\ &\leq |x^+-x^0_s(x)|+|x^0_s(x)-x| \\ &= |f(x,u)-f(x^0_s(x),u^0_s(x))|+|x^0_s(x)-x| \\ &\leq \alpha(\varepsilon)+\varepsilon \end{aligned}$$

Therefore, for a given $\varepsilon>0$, there exists a $k(\varepsilon)$ such that $|x^+-x|\leq\alpha(\varepsilon)+\varepsilon$. Hence, the system converges to a steady state x_∞ and this is such that $x_\infty=x^0_s(x_\infty)\in\mathcal{X}_s$.

Using lemma 6.19 (see the Appendix section of this chapter), it is proved that (x_∞,u_∞) is the optimal steady state of the system, that is $(x_\infty,u_\infty)=(x^*_s,u^*_s)$, where $x^*_s=g_x(y^*_s)$ and $u^*_s=g_u(y^*_s)$.

Finally, the fact that (x^*_s,u^*_s) is a stable equilibrium point for the closed-loop system is proved. That is, for any $\varepsilon>0$ there exists a $\delta>0$ such that for all $|x(0)-x^*_s|\leq\delta$, then $|x(k)-x^*_s|\leq\varepsilon$.

To this aim, define the function $W(x,y_t)=V^0_{N_C,N_P}(x,y_t)-V_O(y^*_s-y_t)$. Then, $W(x^*_s,y_t)=0$. This function is such that $\alpha_W(|x-x^*_s|)\leq W(x,y_t)\leq\beta_W(|x-x^*_s|)$, where α_W and β_W are suitable $\mathcal{K}_\infty$ functions. In fact:

- $W(x,y_t)\geq\alpha_l(|x-x^0_s|)+\alpha_O(|x^0_s-x^*_s|)$. This comes from the fact that the stage cost function is a positive definite function and from the definition of V_O

(assumption 6.3). Hence

$$\begin{aligned} W(x,y_t) &\geq \alpha_W(|x-x_s^0|+|x_s^0-x_s^*|) \\ &\geq \alpha_W(|x-x_s^*|) \end{aligned}$$

- Notice that $V^0_{N_C,N_P}(x,y_t) \leq V_{N_C,N_P}(x,y_s^*)+V_O(y_s^*-y_t)$. Due to the weak controllability of x_s^* (assumption 6.3), there exists a $\mathcal{K}_\infty$ function β_W such that $V_{N_C,N_P}(x,y_s^*) \leq \beta_W(|x-x_s^*|)$. Hence $W(x,y_t) \leq \beta_W(|x-x_s^*|)$.

Then, $\alpha_W(|x(k)-x_s^*|) \leq W(x(k),y_t) \leq W(x(0),y_t) \leq \beta_W(|x-x_s^*|)$ and, hence, $|x(k)-x_s^*| \leq \alpha_W^{-1} \circ \beta_W(|x(0)-x_s^*|)$. So, picking $\delta = \beta_W^{-1} \circ \alpha_W(\varepsilon)$, then $|x(k)-x_s^*| \leq \alpha_W^{-1} \circ \beta_W(\delta) \leq \varepsilon$, proving the stability of x_s^*.

Recapping, it has been proved that for all initial state $x_0 \in \mathcal{X}_{N_C,N_P}$, the closed-loop system converges to an equilibrium point (x_s^*,u_s^*). Moreover, it has been demonstrated that this equilibrium point is stable for the closed-loop system. Therefore, (x_s^*,u_s^*) is an asymptotic stable equilibrium point for the closed-loop system and its domain of attraction is $\mathcal{X}_{N_c,N_p}$. ■

Remark 6.11 *Since $\mathcal{X}_{N_c,N_p} \subseteq \mathcal{X}_{N_c,N_p+1}$, taking $N_p \geq N_c$ the domain of attraction of the controller can be enlarged (Magni et al., 2001a). However, the result of theorem 6.10 and the stability proof are still valid if a formulation with $N=N_c=N_P$ is chosen.*

6.7 Calculation of the terminal ingredients

The conditions for the stabilizing design of the controller require the calculation of a control law capable to locally asymptotically stabilize the system to any steady states contained in a set. This problem is also present in the design of other controllers for tracking, such as the command governors, (Angeli and Mosca, 1999; Bemporad, 1998b; Chisci et al., 2005; Chisci and Zappa, 2003).

A remarkable property of the proposed MPC is that the controller must only stabilize the system locally, and hence a number of existing techniques could be used. The local nature of the obtained controller can be enhanced by using a prediction horizon larger than the control horizon. Next, some practical techniques to cope with this problem are briefly presented.

6.7.1 LTV modeling of the plant in partitions

This method exploits the LTV modeling technique and the partition method proposed in (Wan and Kothare, 2003a,b).

The idea is to design a set of local predictive controllers, whose feasible regions cover the entire steady state manifold. Then, an algorithm is used such that, given a reference y_t and the relative steady state conditions $(x_s(y_s), u_s(y_s))$, we are able to determine the terminal ingredients for the optimization problem $P_{N_c,N_p}(x, y_t)$.

To this aim, consider system (6.1) subject to (6.2). Let $(x_s(y_s), u_s(y_s))$ be a steady condition such that, for any $y_s \in \mathcal{Y}_s$, $x_s(y_s) \in X$ and $u_s(y_s) \in U$.

Choose the sets $B_x = \|x\| \leq \varepsilon_x \in \mathbb{R}^n$ and $B_u = \|u\| \leq \varepsilon_u \in \mathbb{R}^m$ (with ε_x and ε_u typically small) and define

$$\tilde{\mathcal{Y}}_s = \{y_s \in \mathcal{Y}_s : x_s(y_s) \in X - B_x,\ u_s(y_s) \in U - B_u\}.$$

Let $\mathcal{Y}_{s_i}$ be a partition of $\tilde{\mathcal{Y}}_s$, such that $\bigcup_i \mathcal{Y}_{s_i} = \tilde{\mathcal{Y}}_s$. Then a suitable LTV representation of the model function must be found for each region

$$X_i = \bigcup_{y_s \in \mathcal{Y}_{s_i}} x_s(y_s) \oplus B_x, \quad U_i = \bigcup_{y_s \in \mathcal{Y}_{s_i}} u_s(y_s) \oplus B_u$$

For the sake of clarity, consider system (6.1) subject to (6.2), its steady state manifold and a set $\mathcal{Y}_{s_i}$ centered in y_s^i. Since by definition, the steady state manifold is a connected set, we can always find another steady condition $y_s^{i+1} \in \mathcal{Y}_{s_i}$ for which we can determine a set $\mathcal{Y}_{s_{i+1}}$ centered in y_s^{i+1}. Hence we can constructing this sets, until their union covers the steady state manifold (Wan and Kothare, 2003b).

Notice that, since $\mathcal{Y}_{s_i} \subseteq \tilde{\mathcal{Y}}_s$, hence $X_i \subseteq X$ and $U_i \subseteq U$.

For all $x \in X_i$, $u \in U_i$ and $y_s \in \mathcal{Y}_{s_i}$, there exists a LTV representation of system (6.1), that is

$$f(x,u) \quad = \quad f(x_s(y_s), u_s(y_s)) + \sum_{j=1}^{n_i} \lambda_j [A_j(x - x_s(y_s)) + B_j(u - u_s(y_s))] \quad (6.19)$$

where $[A_j\ B_j] \in \{[A_1\ B_1], ..., [A_{n_i}\ B_{n_i}]\}$

By continuity, there exists a control gain $K_i \in \mathbb{R}^{m\times n}$ such that $A_{K_{ij}} = A_j + B_j K_i$ is stable for all j and, moreover, there exists a suitable Lyapunov matrix $P_i \in \mathbb{R}^{n\times n}$ for the LTV, solution of

$$A'_{K_{ij}} P_i A_{K_{ij}} - P_i = -Q - K_i' R K_i$$

for all j.

Define the set $Z_{K_i} = \{z : z \in B_x,\ K_i z \in B_u\}$ and let Ω_i be an invariant set for the LTV (6.19) contained in Z_{K_i}. Then, $\forall y_s \in \mathcal{Y}_{s_i}$ and $x_0 \in x_s(y_s) \oplus \Omega_i$,

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ u(k) &= K_i(x(k) - x_s(y_s)) + u_s(y_s) \end{aligned}$$

such that $x(k) \in X_i \subseteq X$ and $u(k) \in U_i \subseteq U$.

Finally define

$$\Gamma_i = \{(x, y_s) : x \in x_s(y_s) \oplus \Omega_i,\ y_s \in \mathcal{Y}_{s_i}\} = \bigcup_{y_s \in \mathcal{Y}_{s_i}} x_s(y_s) \oplus \Omega_i \times \mathcal{Y}_{s_i}$$

Then, $\forall (x_0, y_s) \in \Gamma_i$, $(x(k), y_s) \in \Gamma_i \subseteq X_i$, $u(k) \in U$ and it can be proved that Γ_i is an admissible invariant set for tracking and that

$$V_{f_i}(x - g_x(y_s), y_s) = (x - g_x(y_s))' P_i (x - g_x(y_s))$$

is a suitable terminal cost function, $\forall y_s \in \mathcal{Y}_{s_i}$.

Hence $\Gamma = \bigcup_i \Gamma_i$ is an invariant set for tracking.

Example 6.12 *Consider a continuous stirred tank reactor (CSTR), (Chisci et al., 2005; Magni et al., 2001a). Assuming constant liquid volume, the CSTR for an exothermic, irreversible reaction, $A \to B$, is described by the following model:*

$$\begin{aligned} \dot{C}_A &= \frac{q}{V}(C_{Af} - C_A) - k_o e^{(\frac{-E}{RT})} C_A \\ \dot{T} &= \frac{q}{V}(T_f - T) - \frac{\Delta H}{\rho C_p} k_o e^{(\frac{-E}{RT})} C_A + \frac{UA}{V\rho C_p}(T_c - T) \end{aligned}$$

where C_A is the concentration of A in the reactor, T is the reactor temperature and T_c is the temperature of the coolant stream. The objective is to regulate $y = x_2 = T$ and $x_1 = C_A$ by manipulating $u = T_c$. Consider also the following constraints on the system: $0 \le C_A \le 1$ mol/l, $280\mathrm{K} \le T \le 370\mathrm{K}$ and $280\mathrm{K} \le T_c \le 370$ K.

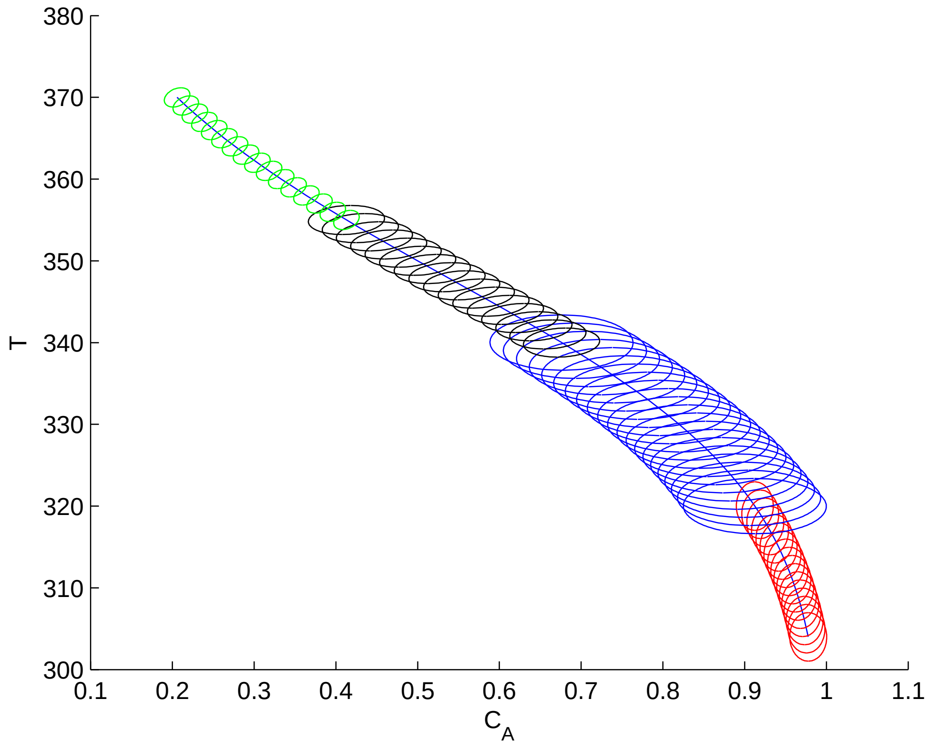


Figure 6.1: Different terminal regions for the CSTR.

To the aim of calculating the terminal ingredients of the nonlinear MPC for tracking with the method introduced above, the steady state manifold of the system has been divided in 4 partitions, given by $\mathcal{Y}_{s_1} = [304.17; 320]$, $\mathcal{Y}_{s_2} = [320; 340]$, $\mathcal{Y}_{s_3} = [340; 355]$ *and* $\mathcal{Y}_{s_4} = [355; 370]$ *respectively. The projection of the obteined regions in the state space are show in figure 6.1. Notice how the regions calculated for the four partitions, cover the entire steady state manifold. The union of these regions provides the invariant set for tracking to be used as terminal constraints.*

6.7.1.1 Implementation

The construction of the terminal ingredients described so far is an off-line task, during which we store in a look-up table the different ingredients $\mathcal{Y}_{s_{i(y_s)}}$, $P_{i(y_s)}$, Ω_i. On-line, we choose a terminal condition given a reference y_t.

1. For a given reference y_t determine y_s, and then determine $i(y_s)$ such that $y_s \in$

$y_{s_{i(y_s)}}$.

2. Choose the terminal cost $V_f(x(N_p) - x_s(y_s), y_s)$ as:

$$V_f(x(N_p) - x_s(y_s), y_s) = (x(N_p) - x_s(y_s))' P_{i(y_s)} (x(N_p) - x_s(y_s))$$

3. Choose the terminal constraint as:

$$x(N_p) - x_s(y_s) \in \Omega_i$$

Remark 6.13 *A drawback of this approach is that the size of Ω_i could be very small. However, taking $N_p \gg N_c$ the domain of attraction of the controller can be enlarged (Magni et al., 2001a).*

6.7.2 Feedback linearization

There exist some classes of model functions that allow to find a suitable feedback aimed to linearize (Khalil, 1996; Isidori, 1995) or pseudolinearize (Findeisen et al., 2000; Rakovic et al., 2006)the plant. The feedback linearization approach consists in finding a transformation of the nonlinear system using a suitable control action, given by the feedback, and such a change of variables. This approach can be applied to systems in the form

$$\begin{aligned} x^+ &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, $y \in \mathbb{R}^p$ is the controlled output and x^+ is the successor state. The aim is to design a control input:

$$u = \alpha(x) + \beta(x)v$$

which provides a linear transformation from the new input v to the output.

The aim of feedback linearization, is, hence, to rewrite the system in such a form that the states of the new system are the output y and its $n-1$ derivatives, obtained using the Lie derivative. The number of time the original system has to be differentiated to let the input u appear as a linear term, is called relative dergree of the system (Khalil, 1996; Isidori, 1995). Once the plant is represented by a linear model, the ingredients can be calculated as proposed in (Limon et al., 2008a) and (Ferramosca et al., 2009a) for linear systems.

6.8 MPC for tracking without terminal constraint

The stabilizing design of MPC based on terminal ingredients requires the calculation of the terminal control law, the terminal cost function and the terminal region. While the calculation of the first two ingredients can be done by efficient techniques, the calculation of the terminal region may be cumbersome. Fortunately, in (Limon et al., 2006b) it has been proved that the terminal constraint can be removed by a suitable weighting of the terminal cost function. In fact, a larger weighting factor implies a larger domain of attraction of the predictive controller without the terminal constraint. In this section it is demonstrated that this result can be also applied to the MPC for tracking.

Consider that $V_f(x - x_s, y_s)$ is a terminal cost function and $\kappa(x, y_s)$ a terminal control law that satisfy assumption 6.9. Define the region Γ_α as follows

$$\Gamma_\alpha = \{(x, y_s) : V_f(x - g_x(y_s), y_s) \leq \alpha\} \tag{6.20}$$

and assume that Γ_α is an invariant set for tracking.

Let define $P^\gamma_{N_c,N_p}(x, y_t)$ the optimization problem resulting from removing the terminal constraint $(x(N_p), y_s) \in \Gamma$ and taking the weighted cost $\gamma V_f(x - x_s, y_s)$ as terminal cost function in the optimization problem $P_{N_c,N_p}(x, y_t)$.

The following results are the extension of (Limon et al., 2006b). In the sequel it is assumed that the hypothesis of theorem 6.10 hold.

Lemma 6.14 *Let $x^{\gamma,0}(j; x, y_t)$ be the optimal trajectory of the optimization problem $P^\gamma_{N_c,N_p}(x, y_t)$ for any $\gamma \geq 1$.*
If $x^{\gamma,0}(N_p; x, y_t) \notin \Gamma_\alpha$, then $x^{\gamma,0}(j; x, y_t) \notin \Gamma_\alpha$ for all $j = 0, \cdots, N_p$.

The proof of this lemma is given in (Limon et al., 2006b, Lemma 1).

Lemma 6.15 *Let $d = \alpha_\ell(\beta_f^{-1}(\alpha))$, then $\ell(x - g_x(y_s), u - g_u(y_s)) \geq d$ for all $(x, y_s) \notin \Gamma_\alpha$.*

Let $V^{\gamma,0}_{N_c,N_p}(x, y_t)$ be the optimal cost of $P^\gamma_{N_c,N_p}(x, y_t)$ and let define the following level set

$$\hat{\Upsilon}_{N_p,\gamma}(y_t) = \{x : V^{\gamma,0}_{N_c,N_p}(x, y_t) \leq \ell(x - g_x(y_s), k(x, y_s) - g_u(y_s)) + (N_p - 1)d + \alpha\} \tag{6.21}$$

Then we can state the following theorem

Theorem 6.16 *Let $\kappa^{\gamma}_{N_c,N_p}(x, y_t)$ be the predictive control law derived from $P^{\gamma}_{N_c,N_p}(x, y_t)$ for any $\gamma \geq 1$. Then for all $x(0) \in \hat{\Upsilon}_{N_p,\gamma}(y_t)$, the system controlled by $\kappa^{\gamma}_{N_c,N_p}(x, y_t)$ is stable, converges to an equilibrium point, fulfils the constraints along the time and besides*

(i) If $y_t \in \mathcal{Y}_s$ then $\lim_{k\to\infty} \|y(k) - y_t\| = 0$.

(ii) If $y_t \notin \mathcal{Y}_s$, then $\lim_{k\to\infty} \|y(k) - y_s^*\| = 0$, where

$$y_s^* = \arg \min_{y_s \in \mathcal{Y}_s} V_O(y_s - y_t)$$

Proof: First it is proved that for any $x \in \hat{\Upsilon}_{N_p,\gamma}(y_t)$ the optimal solution of the MPC problem satisfies the terminal constraint. From Lemma 6.14 it is inferred that if the terminal region is not reached, then all the trajectory of the systems is out of Γ_α and hence

$$V^{\gamma,0}_{N_c,N_p}(x, y_t) > \ell(x - g_x(y_s), k(x, y_s) - g_u(y_s)) + (N_p - 1)d + \alpha$$

which implies that $x \notin \hat{\Upsilon}_{N_p,\gamma}(y_t)$, which is a contradiction. Therefore for any $x \in \hat{\Upsilon}_{N_p,\gamma}(y_t)$, we have that the optimal solution of the MPC satisfies the terminal constraint.

Now, it is proved that $\hat{\Upsilon}_{N_p,\gamma}(y_t)$ is an invariant set for the closed-loop system. Consider that $x \in \hat{\Upsilon}_{N_p,\gamma}(y_t)$, then $x^{\gamma,0}(N_p; x, y_t) \in \Gamma_\alpha$. This fact plus the assumptions of theorem 6.10 make the monotonicity property of the optimal cost holds. Hence:

$$\begin{aligned} V^{\gamma,0}_{N_c,N_p}(x^{\gamma,0}(1; x, y_t), y_t) &\leq V^{\gamma,0}_{N_c,N_p-1}(x, y_t) = V^{\gamma,0}_{N_c,N_p}(x, y_t) - \ell(x - g_x(y_s), k(x, y_s) - g_u(y_s)) \\ &\leq (N_p - 1)d + \alpha \end{aligned}$$

Asymptotic stability of the closed-loop system is proved following the same arguments as in the proof of of theorem 6.10. ■

Using similar arguments as in (Limon et al., 2006b), it is easy to show that the set

$$\Upsilon_{N_p,\gamma}(y_t) = \{x : V^{\gamma,0}_{N_c,N_p}(x, y_t) \leq N_p d + \alpha\}$$

is also a domain of attraction of the proposed controller. Notice that, using this result, an explicit expression of the optimal cost is not required.

It is clear that taking a larger prediction horizon N_p implies an enlargement of the region of attraction $\Upsilon_{N_p,\gamma}(y_t)$. Furthermore, as in (Limon et al., 2006b), a larger value of γ implies also an enlargement of $\Upsilon_{N_p,\gamma}(y_t)$. The following property can be derived:

Property 6.17 *Assume that X and U are compact sets. Let D be a constant such that $\ell(x-x_s, u-u_s) \leq D$ for all $x \in X$, $u \in U$ and $(x_s, u_s) \in Z_s$. Let $\hat{V}_O$ be a constant such that $V_O(y_s - y_t) \leq \hat{V}_O$ for all $y_s \in \mathcal{Y}_s$ and for all possible y_t. Let be $\gamma_0 = \frac{N_p(D-d)+\hat{V}_O}{\alpha}$. Then*

(i) $\Upsilon_{N_p,\gamma_0}(y_t)$ contains the domain of attraction of the controller with equality constraint, i.e. $\mathcal{X}_{N_c}$

(ii) For any $\gamma \geq \gamma_0$, the set $\Upsilon_{N_p,\gamma_0}(y_t)$ contains the domain of attraction of the controller with the terminal constraint $\Gamma_{\rho\alpha}$, where $\rho = 1 - \frac{\gamma_0}{\gamma}$.

6.9 Illustrative example

In this section, the three formulation of the nonlinear MPC for tracking presented in this chapter, are tested on the 4 tanks process, described in the Appendix A.
The nonlinear continuous time model of the quadruple tank process system (Johansson, 2000) can be derived from first principles as follows

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_a}{A_1}q_a && (6.22)\\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_b}{A_2}q_b && (6.23)\\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_b)}{A_3}q_b\\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_a)}{A_4}q_a
\end{aligned}$$

See chapter 2 for details. The nonlinear discrete time model of system (6.9) is obtained by discretizing equation (6.9) using a 5-th order Runge-Kutta method and taking as sampling time $Ts = 5$s.

First, the controller with terminal equality constraint is presented. In the next section, the nonlinear MPC for tracking using terminal inequality constraint is tested.

Finally, another test to show the properties of the MPC without terminal constraint is presented.

All the simulations have been run in $MATLAB^{®}$ 7.10, using the *fmincon* function for the optimizations.

6.9.1 Equality terminal constraint

The objective of this test is to show how the nonlinear MPC for tracking works, in case of equality terminal constraint. In the test, five references have been considered:$y_{t,1} = (0.3, 0.3)$, $y_{t,2} = (1.25, 1.25)$, $y_{t,3} = (0.35, 0.8)$, $y_{t,4} = (1, 0.8)$ and $y_{t,5} = (h_1^0, h_2^0)$. Notice that $y_{t,3}$ is not an equilibrium output for the system. The initial state is $x_0 = (0.65, 0.65, 0.6658, 0.6242)$. Two tests have been run, considering an MPC with $N = 3$ and $N = 15$ respectively. The weighting matrices have been chosen as $Q_y = I_2$, $Q = C'Q_yC + 0.01I_4$ and $R = 0.01I_2$. Notice that

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The offset cost function has been chosen as $V_O = \alpha\|y_s - y_t\|_\infty$, whith $\alpha = 100$.

In Figure 6.2 and 6.3 the time evolution of h_1, h_2, h_3, h_4, q_a and q_b is depicted.

The evolutions of the references, the outputs and the artificial references are drawn respectively in dashed-dotted, solid and dashed line. See that the controller always steers the system to the desired setpoint, whenever it is admissible. When the target is not an admissible output ($y_{t,3}$), the controller steers the system to the point that minimize the offset cost function.

In Figure 6.4 and 6.5 the time evolution of h_1, h_2, h_3, h_4, q_a and q_b is depicted.

The evolutions of the references is drawn in dashed-dotted line while the outputs and the artificial references are drawn respectively in solid and dashed line. As in the previous test, the controller always steers the system to the desired setpoint, when this point is admissible. In case of not admissible target ($y_{t,3}$), the controller steers the system to the point that minimizes the offset cost function. Notice that, due to the larger horizon, the controller is faster than the previous one.

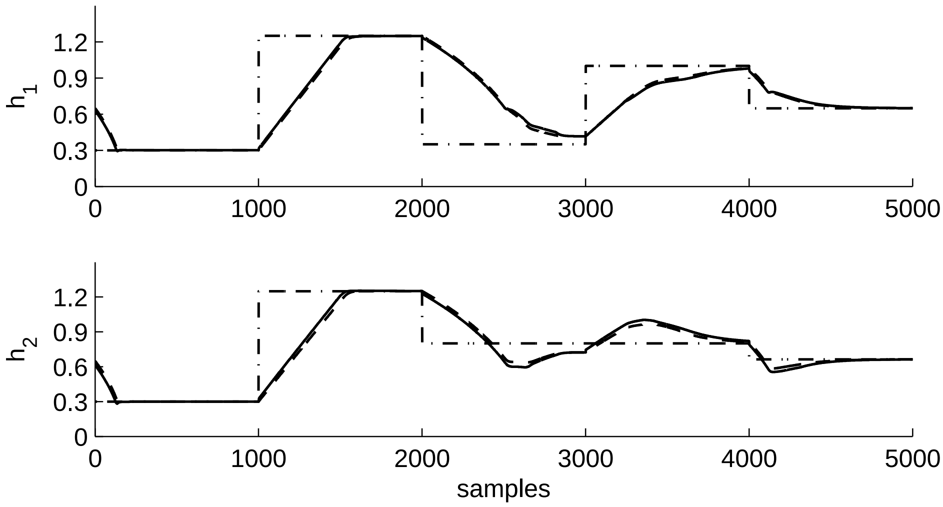


Figure 6.2: Evolution of h_1 and h_2 with $N = 3$.

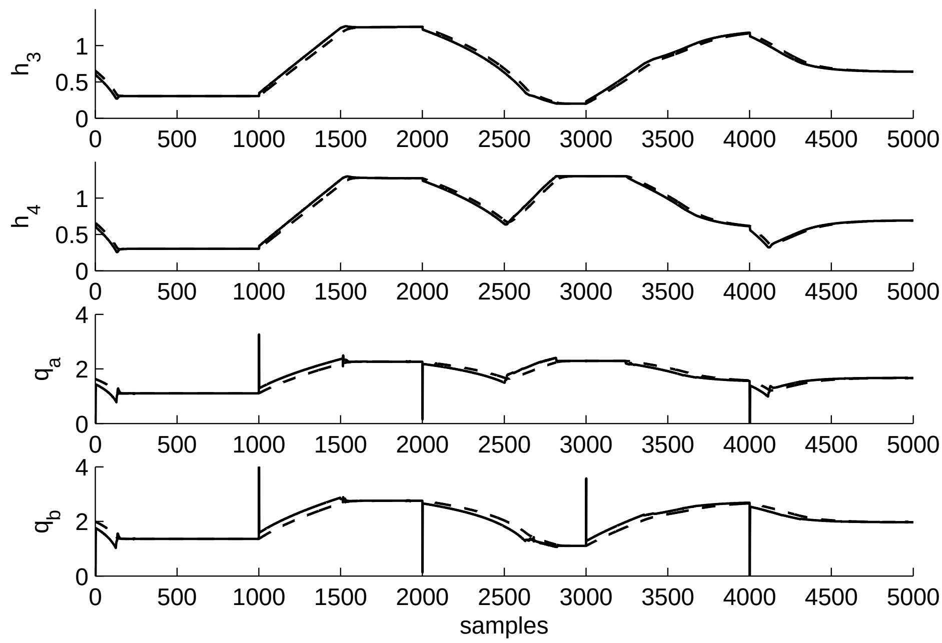


Figure 6.3: Evolution of h_3, h_4 q_a and q_b with $N = 3$.

6.9.2 Inequality terminal constraint

In this section, a test that show the properties of the nonlinear MPC for tracking with inequality terminal constraint is presented. In particular, the terminal ingredients of

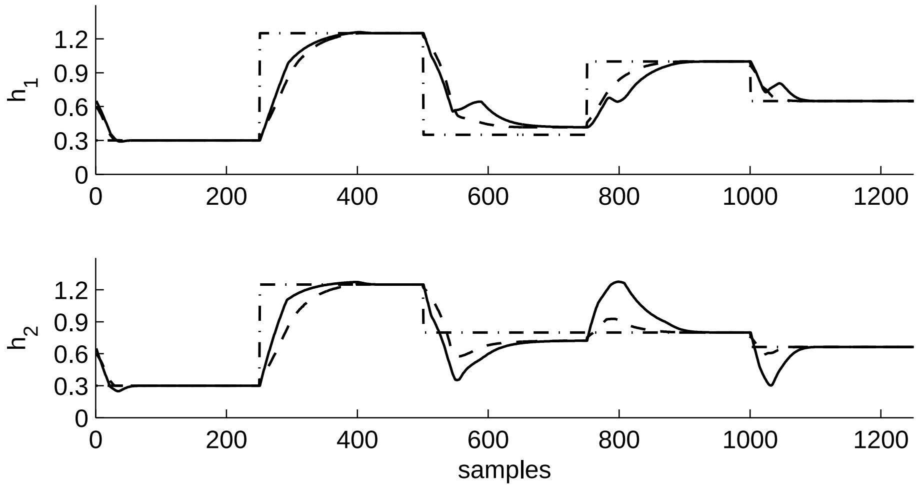


Figure 6.4: Evolution of h_1 and h_2 with $N = 15$.

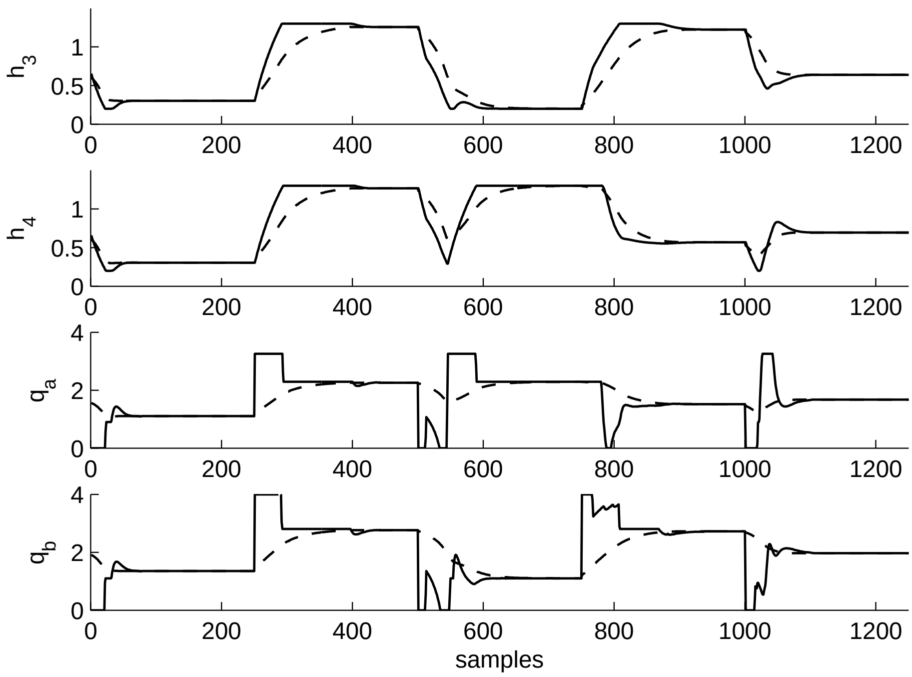


Figure 6.5: Evolution of h_3, h_4 q_a and q_b with $N = 15$.

the controller, have been calculated using the method presented in section 6.7.1.

To this aim, the space of equilibrium points, has been partitioned in subspaces, based on the geometry of set $\mathcal{Y}_s$. Based on this, seven different regions of equilibrium points have been obtained. For any region, the static gain K and the matrix P have been calculated solving some LMIs. The seven region obtained and the relative values of K and P are the following:

- $\mathcal{Y}_{s_1} = \{y_s : (0.2; 0.2) \leq y_s \leq (0.42; 0.53)\}$

$$K_1 = \begin{bmatrix} 0.0267 & 0.1213 & 0.1633 & -0.3289 \\ 0.2401 & 0.1154 & -0.2035 & 0.3473 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 17.7433 & 1.2019 & 12.2024 & 1.8258 \\ 1.2019 & 16.9910 & 1.5558 & 11.4664 \\ 12.2024 & 1.5558 & 23.7045 & 4.2748 \\ 1.8258 & 11.4664 & 4.2748 & 26.9804 \end{bmatrix}$$

- $\mathcal{Y}_{s_2} = \{y_s : (0.2; 0.53) \leq y_s \leq (0.42; 0.72)\}$

$$K_2 = \begin{bmatrix} 0.0731 & 0.1902 & 0.2881 & -0.1673 \\ 0.1361 & -0.1086 & -0.1398 & 0.2884 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 17.2518 & 0.4228 & 11.5155 & 1.3011 \\ 0.4228 & 17.2691 & 0.1990 & 7.9675 \\ 11.5155 & 0.1990 & 18.3620 & 3.0846 \\ 1.3011 & 7.9675 & 3.0846 & 12.3569 \end{bmatrix}$$

- $\mathcal{Y}_{s_3} = \{y_s : (0.42; 0.2) \leq y_s \leq (0.87; 0.53)\}$

$$K_3 = \begin{bmatrix} -0.0837 & 0.1479 & 0.1109 & -0.2378 \\ 0.1433 & 0.1627 & -0.3395 & 0.3921 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 25.0142 & 1.3224 & 16.9693 & 1.7039 \\ 1.3224 & 17.5763 & 1.8949 & 12.8116 \\ 16.9693 & 1.8949 & 31.7671 & 4.9483 \\ 1.7039 & 12.8116 & 4.9483 & 28.2461 \end{bmatrix}$$

- $\mathcal{Y}_{s_4} = \{y_s : (0.42; 0.53) \leq y_s \leq (0.87; 0.72)\}$

$$K_4 = \begin{bmatrix} -0.0632 & 0.1489 & 0.0746 & -0.2851 \\ 0.1471 & 0.1015 & -0.3714 & 0.3300 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 26.6981 & 1.8242 & 21.7313 & 2.4324 \\ 1.8242 & 21.0805 & 2.3765 & 17.4553 \\ 21.7313 & 2.3765 & 50.4088 & 6.2126 \\ 2.4324 & 17.4553 & 6.2126 & 45.0831 \end{bmatrix}$$

- $\mathcal{Y}_{s_5} = \{y_s : (0.42; 0.72) \leq y_s \leq (0.87; 1.30)\}$

$$K_5 = \begin{bmatrix} -0.0225 & 0.2117 & 0.1708 & -0.1772 \\ 0.0673 & -0.0502 & -0.3587 & 0.2684 \end{bmatrix}$$

$$P_5 = \begin{bmatrix} 26.1159 & 0.4025 & 20.7702 & 1.5762 \\ 0.4025 & 22.3717 & -0.7066 & 13.4132 \\ 20.7702 & -0.7066 & 45.0959 & 3.8375 \\ 1.5762 & 13.4132 & 3.8375 & 22.0282 \end{bmatrix}$$

- $\mathcal{Y}_{s_6} = \{y_s : (0.87; 0.53) \leq y_s \leq (1.30; 0.72)\}$

$$K_6 = \begin{bmatrix} -0.1317 & -0.0301 & 0.0873 & -0.3848 \\ 0.0266 & 0.1371 & -0.0453 & 0.6802 \end{bmatrix}$$

$$P_6 = \begin{bmatrix} 21.8742 & -0.7686 & 9.3239 & -1.6253 \\ -0.7686 & 19.5727 & 0.2713 & 14.0839 \\ 9.3239 & 0.2713 & 11.5076 & 1.7596 \\ -1.6253 & 14.0839 & 1.7596 & 25.8846 \end{bmatrix}$$

- $\mathcal{Y}_{s_7} = \{y_s : (0.87; 0.72) \leq y_s \leq (1.30; 1.30)\}$

$$K_7 = \begin{bmatrix} -0.0957 & 0.0075 & 0.1574 & -0.3348 \\ 0.0834 & -0.0168 & -0.1525 & 0.4590 \end{bmatrix}$$

$$P_7 = \begin{bmatrix} 26.7931 & -0.7498 & 14.7933 & -0.9438 \\ -0.7498 & 24.2919 & -0.7085 & 15.2670 \\ 14.7933 & -0.7085 & 20.1643 & 1.9967 \\ -0.9438 & 15.2670 & 1.9967 & 25.7673 \end{bmatrix}$$

Moreover, to counteract the small dimension of the ellipsoids, a prediction horizon N_p larger than the control horizon N_c, has been chosen. In particular the setup of the controller has been the following: $N_c = 3$, $N_p = 20$, $Q_y = I_2$, $Q = C'Q_yC + 0.01I_4$, $R = 0.01I_2$. Notice that

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and that the offset cost function has been chosen as $V_O = \alpha\|y_s - y_t\|_\infty$, whith $\alpha = 100$. As in the previous test, five references have been considered:$y_{t,1} = (0.3, 0.3)$, $y_{t,2} = (1.25, 1.25)$, $y_{t,3} = (0.35, 0.8)$, $y_{t,4} = (1, 0.8)$ and $y_{t,5} = (h_1^0, h_2^0)$, and the point $x_0 = (0.65, 0.65, 0.6658, 0.6242)$ has been chosen as initial state.

In Figure 6.6 and 6.7 the time evolution of h_1, h_2, h_3, h_4, q_a and q_b is depicted.

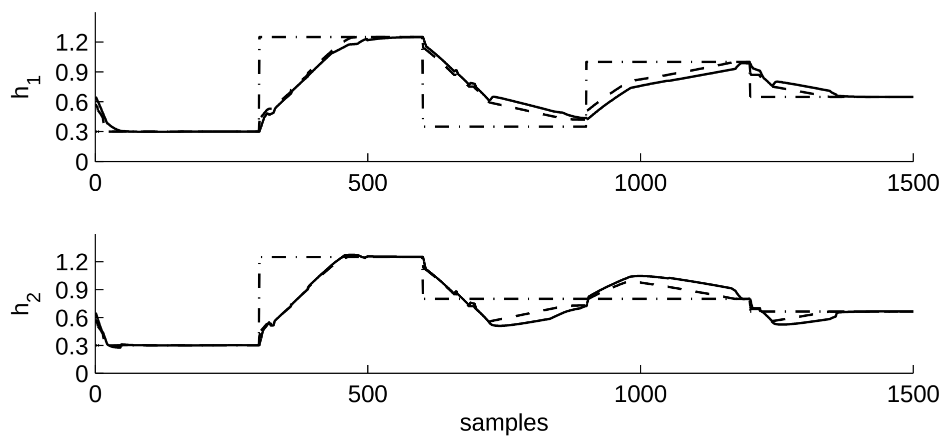


Figure 6.6: Evolution of h_1 and h_2 in case of inequality terminal constraint.

The evolutions of the references is drawn in dashed-dotted line while the outputs and the artificial references are drawn respectively in solid and dashed line. As in the previous testS, the controller always steers the system to the desired setpoint, when this point is admissible. When the target is a not adimissible setpoint, $(y_{t,3})$, the controller steers the system to the point that minimize the offset cost function. Notice that, the controller sometimes seems to loose continuity. These *jumps* are due to the fact that the controller moves from a terminal ingredient to another.

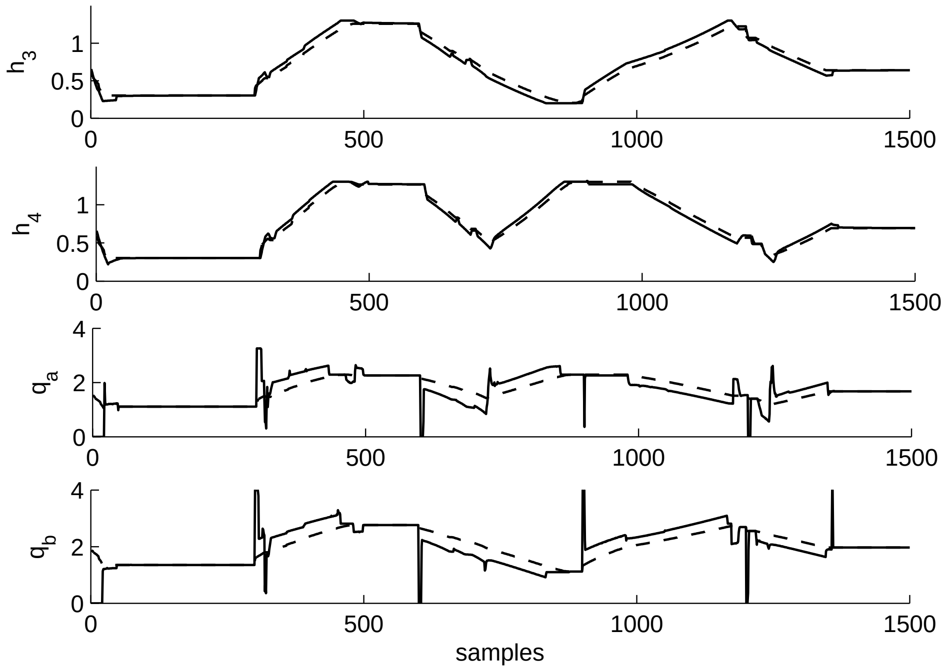


Figure 6.7: Evolution of h_3, h_4 q_a and q_b in case of inequality terminal constraint.

6.10 Domains of attraction

In this section, a comparison of the domains of attraction of the controllers used for the simulations of the previous tests, is presented. In case of nonlinear systems, it is not easy to exactly calculate the domains of attraction using the set control theory, as in case of linear systems. For this reason, only estimations of these sets can be done. The estimation of the domains of attraction presented in this section has been carried out solving the so-called *Phase I* problem (Boyd and Vandenberghe, 2006, pag. 579). In particular, a series of *Phase I* problems has been solved for a grid of points $x = (h_1, h_2, h_3, h_4)$, considering the constraints of the MPC problems as the set of inequalities and equalities.

The results obtained are that, the two domain of attraction extimated for the MPC with terminal equality constraint are $\mathcal{X}_3$ with Chebichev's radius $r_3 = 0.0518$ for $N = 3$, and $\mathcal{X}_{15}$ with Chebichev's radius $r_{15} = 0.1841$ for $N = 15$. The domain of attraction extimated for the MPC with inequality terminal constraint is a set $\mathcal{X}_{3,20}$ with Chebichev's radius $r_{3,20} = 0.4546$. The results are also shown in fig 6.8. These sets represent the

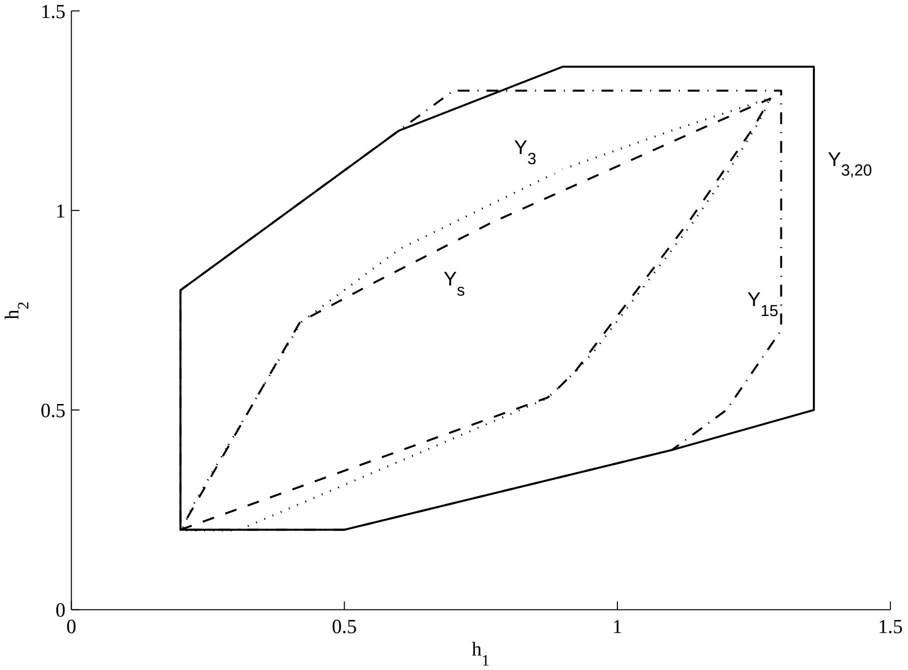


Figure 6.8: Domains of attraction of the controllers proposed in this example.

projection onto y of the domanins of attractions extimated. The set of admissible steady output $\mathcal{Y}_s$ is drawn in dashed line. The projections of the domains of attraction of the MPC with equality terminal constraints are drawn respectively in dotted line ($\mathcal{Y}_3$) for $N = 3$ and in dashed-dotted line ($\mathcal{Y}_{15}$) for $N = 15$. See that, as obvious, the dimension of the feasible set grows with N. The projection of the domain of attraction of the MPC controller with inequality terminal constraint, $\mathcal{Y}_{3,20}$ is plotted in solid line.

6.11 Conclusions

In this chapter the MPC for tracking for constrained nonlinear system has been presented. This controller has been presented in three different formulation: with equality terminal constraint, with inequality terminal constraint and without terminal constraint.
Under some assumptions, stability of the controller has been proved for the three different formulation presented. The controller steers the system to the target if this is admissible. If not, the controller converges to an admissible steady state according to

the offset cost function.

6.12 Appendix

In this Appendix section, the technical lemmas used to prove Theorems 6.5 and 6.10 are presented. In particular, this lemmas prove the optimality of the steady state.

Lemma 6.18 *Consider system* (6.1) *subject to constraints* (6.2). *Consider that assumption 6.3 holds. Consider a given target* y_t *and assume that for a given state* x *the optimal solution of* $P_N(x, y_t)$ *is such that* $x_s^0(x, y_t) = g_x(y_s^0(x, y_t))$ *and* $u_s^0(x, y_t) = g_u(y_s^0(x, y_t))$. *Let* $\tilde{y}_s \in \mathcal{Y}_s$ *be given by*

$$\tilde{y}_s \triangleq \arg \min_{y_s \in \lambda \mathcal{Y}_s} V_O(y_s - y_t)$$

Then

$$x_s^0(x, y_t) = \tilde{x}_s, \; u_s^0(x, y_t) = \tilde{u}_s, \; y_s^0(x, y_t) = \tilde{y}_s$$

Proof: Consider that the optimal solution of $P_N(x, y_t)$ is (x_s^0, u_s^0, y_s^0)[2]. The optimal cost function is $V_N^0(x, y_t) = V_O(y_s^0 - y_t)$.

The lemma will be proved by contradiction. Assume that $y_s^0 \neq \tilde{y}_s$.

Define $\hat{y}_s$ given by

$$\hat{y}_s = \beta y_s^0 + (1 - \beta)\tilde{y}_s \quad \beta \in [0, 1]$$

Assuming $\mathcal{Y}_s$ convex, this point is an admissiblo steady state. Therefore, defining as $\mathbf{u}$ the sequence of control actions $\mathbf{u} = \{\hat{u}_s, ..., \hat{u}_s\}$, it is easily inferred that $(\mathbf{u}, \hat{y}_s)$ is a feasible solution for $P_N(x_s^0, y_t)$. Then using standard procedures in MPC, we have

[2]In this proof, the dependence of the optimal solution from (x, y_t) will be omitted for the sake of clarity.

that

$$\begin{aligned} V_N^0(x_s^0, y_t) &= V_O(y_s^0 - y_t) \\ &\leq V_N(x_s^0, y_t; \mathbf{u}, \hat{y}_s) \\ &= \sum_{j=0}^{N-1} \ell((x(j) - \hat{x}_s), (u(j) - \hat{u}_s)) + V_O(\hat{y}_s - y_t) \\ &= V_O(\hat{y}_s - y_t) \end{aligned}$$

Define $W(x_s^0, y_t, \beta) \triangleq V_O(\hat{y}_s - y_t)$ and notice that $W(x_s^0, y_t, \beta) = V_N^0(x_s^0, y_t)$ for $\beta = 1$. Taking the partial of W about β we have that

$$\frac{\partial W}{\partial \beta} = g'(y_s^0 - \tilde{y}_s)$$

where $g' \in \partial V_O(\hat{y}_s - y_t)$, defining $\partial V_O(\hat{y}_s - y_t)$ as the subdifferential of $V_O(\hat{y}_s - y_t)$. Evaluating this partial for $\beta = 1$ we obtain that:

$$\left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} = g^{0'}(y_s^0 - \tilde{y}_s)$$

where $g^{0'} \in \partial V_O(y_s^0 - y_t)$, defining $\partial V_O(y_s^0 - y_t)$ as the subdifferential of $V_O(y_s^0 - y_t)$. Taking into account that V_O is a subdifferentiable function, from convexity (Boyd and Vandenberghe, 2006) we can state for every y_s^0 and $\tilde{y}_s$ that

$$g^{0'}(y_s^0 - \tilde{y}_s) \geq V_O(y_s^0 - y_t) - V_O(\tilde{y}_s - y_t)$$

Taking into account that $y_s^0 \neq \tilde{y}_s$, $V_O(y_s^0 - y_t) - V_O(\tilde{y}_s - y_t) > 0$, it can be derived that

$$\left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} \geq V_O(y_s^0 - y_t) - V_O(\tilde{y}_s - y_t) > 0$$

This means that there exists a $\beta \in [\hat{\beta}, 1)$ such that $W(x_s^0, y_t, \beta)$ is smaller than the value of $W(x_s^0, y_t, \beta)$ for $\beta = 1$, which equals to $V_N^0(x_s^0, y_t)$.

This contradicts the optimality of the solution and hence the result is proved. ■

Lemma 6.19 *Consider system* (6.1) *subject to constraints* (6.2). *Consider that assumption 6.3 and 6.9 hold. Consider a given target y_t and assume that for a given* *state x the optimal solution of $P_{N_c,N_p}(x,y_t)$ is such that $x_s^0(x,y_t) = g_x(y_s^0(x,y_t))$ and $u_s^0(x,y_t) = g_u(y_s^0(x,y_t))$. Let $\tilde{y}_s \in \lambda\mathcal{Y}_s$ be given by*

$$\tilde{y}_s \triangleq \arg\min_{y_s\in\lambda\mathcal{Y}_s} V_O(y_s - y_t)$$

Then

$$x_s^0(x,y_t) = \tilde{x}_s,\ u_s^0(x,y_t) = \tilde{u}_s,\ y_s^0(x,y_t) = \tilde{y}_s$$

Proof: Consider that the optimal solution of $P_{N_c,N_p}(x,y_t)$ is (x_s^0, u_s^0, y_s^0)[3]. The optimal cost function is $V^0_{N_c,N_p}(x,y_t) = V_O(y_s^0 - y_t)$.

The lemma will be proved by contradiction. Assume that $y_s^0 \neq \tilde{y}_s$.

Define $\hat{y}_s$ given by

$$\hat{y}_s = \beta y_s^0 + (1-\beta)\tilde{y}_s \quad \beta \in [0,1]$$

From continuity arguments it can be derived that there exists a $\hat{\beta} \in [0,1)$ such that for every $\beta \in [\hat{\beta}, 1)$, $(x_s^0, \hat{y}_s) \in \Gamma$.

Therefore, defining as $\mathbf{u}$ the sequence of control actions derived from the control law $k(x,\hat{y}_s)$, it is easily inferred that $(\mathbf{u}, \hat{y}_s)$ is a feasible solution for $P_{N_c,N_p}(x_s^0, y_t)$. Then from assumption 6.9 and using standard procedures in MPC, we have that

$$\begin{aligned}
V^0_{N_c,N_p}(x_s^0,y_t) &= V_O(y_s^0 - y_t)\\
&\leq V_{N_c,N_p}(x_s^0, y_t; \mathbf{u}, \hat{y}_s)\\
&= \sum_{i=0}^{N_p-1} \ell((x(i)-\hat{x}_s),(k(x(i),\hat{y}_s)-\hat{u}_s)) + V_f(x(N_p)-\hat{x}_s,\hat{y}_s) + V_O(\hat{y}_s - y_t)\\
&\leq V_f(x_s^0 - \hat{x}_s, \hat{y}_s) + V_O(\hat{y}_s - y_t)\\
&\leq L_{V_f}\|y_s^0 - \hat{y}_s\|^\sigma + V_O(\hat{y}_s - y_t)\\
&= L_{V_f}(1-\beta)^\sigma \|y_s^0 - \tilde{y}_s\|^\sigma + V_O(\hat{y}_s - y_t)
\end{aligned}$$

where $L_{V_f} = L_g^\sigma b$ and L_g is the Lipshitz constant of $g_x(\cdot)$.

Define $W(x_s^0, y_t, \beta) \triangleq L_{V_f}(1-\beta)^\sigma\|y_s^0 - \tilde{y}_s\|^\sigma + V_O(\hat{y}_s - y_t)$ and notice that $W(x_s^0, y_t, \beta) = V^0_{N_c,N_p}(x_s^0, y_t)$ for $\beta = 1$. Taking the partial of W about β we have that

$$\frac{\partial W}{\partial \beta} = -L_{V_f}\sigma(1-\beta)^{\sigma-1}\|y_s^0 - \tilde{y}_s\|^\sigma + g'(y_s^0 - \tilde{y}_s)$$

[3] In this proof, the dependence of the optimal solution from (x, y_t) will be omitted for the sake of clarity.

where $g' \in \partial V_O(\hat{y}_s - y_t)$, defining $\partial V_O(\hat{y}_s - y_t)$ as the subdifferential of $V_O(\hat{y}_s - y_t)$. Evaluating this partial for $\beta = 1$ we obtain that:

$$\left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} = g^{0'}(y_s^0 - \tilde{y}_s)$$

where $g^{0'} \in \partial V_O(y_s^0 - y_t)$, defining $\partial V_O(y_s^0 - y_t)$ as the subdifferential of $V_O(y_s^0 - y_t)$. Taking into account that V_O is a subdifferentiable function, from convexity (Boyd and Vandenberghe, 2006) we can state for every y_s^0 and $\tilde{y}_s$ that

$$g^{0'}(y_s^0 - \tilde{y}_s) \geq V_O(y_s^0 - y_t) - V_O(\tilde{y}_s - y_t)$$

Taking into account that $y_s^0 \neq \tilde{y}_s$, $V_O(y_s^0 - y_t) - V_O(\tilde{y}_s - y_t) > 0$, it can be derived that

$$\left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} \geq V_O(y_s^0 - y_t) - V_O(\tilde{y}_s - y_t) > 0$$

This means that there exists a $\beta \in [\hat{\beta}, 1)$ such that $W(x_s^0, y_t, \beta)$ is smaller than the value of $W(x_s^0, y_t, \beta)$ for $\beta = 1$, which equals to $V^0_{N_c,N_p}(x_s^0, y_t)$.

This contradicts the optimality of the solution and hence the result is proved. ■

CHAPTER 7

Economic MPC for a changing economic criterion

7.1 Introduction

This chapter is dedicated to the problem of the Model Predictive Control based on economic cost functions.

The standard procedures in all industrial advanced process control systems is to decompose a plant's economic optimization in two levels. The first level performs a steady state optimization, and it is usually called as Real Time Optimizer (RTO). The RTO determines the optimal setpoints and sends them to the second level, the advanced control systems, which performs a dynamic optimization. In many control process, MPC is used as the advanced control formulation chosen for this level (Rawligns and Amrit, 2009).

In (Rawlings et al., 2008) the authors consider the problem of a setpoint that becomes unreachable due to the system constraints. The usual method to handle this problem is to transform the unreachable setpoint into a reachable steady state target using a separate steady state optimization. This paper proposes an alternative approach in which the unreachable setpoint is retained in the controller's stage cost and objective function. The use of this objective function induces an interesting fast/slow asymmetry in the system's tracking response that depends on the system initial condition, speeding up approaches to the unreachable setpoint, but slowing down departures from the unreachable setpoint. In (Rawligns and Amrit, 2009) the authors consider the case of replacing the setpoint objective function with an objective measuring some economic performance. In (Diehl et al., 2011) the authors also show that the economic MPC schemes admit a Lyapunov function to establish stability properties.

If the economic criterion changes, for instance due to changes in the prices, expected demand, etc., the economically optimal admissible steady state where the controller steers the system may change, and the feasibility of the controller may be lost. This loss of feasibility recall the tracking problem treated in (Limon et al., 2008a; Ferramosca et al., 2009a) for linear systems and in (Limon et al., 2009b) for nonlinear systems, and in the previous chapters of this thesis. These controllers ensure that under any change of the setpoint, the closed-loop system maintains the feasibility of the controller, ensures the convergence to the setpoint if admissible (and the closest steady state if it is not), and inherits the optimality property of the MPC for regulation.

In this chapter, an economic MPC for a changing economic criterion is presented. This controller inherits the feasibility guarantee of the MPC for tracking (Limon et al., 2008a; Ferramosca et al., 2009a) and the optimality of the economic MPC (Rawlings et al., 2008; Diehl et al., 2011). A particular stage cost function is proposed for establishing asymptotic stability.

7.2 Problem statement

Consider a system described by a nonlinear time-invariant discrete time model

$$x^+ \;=\; f(x,u) \tag{7.1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector and x^+ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u})$, $j \in \mathbb{I}_{\geq 0}$, where $x = \phi(0; x, \mathbf{u})$. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively.

The system is subject to hard constraints on state and input:

$$x(k) \in X, \quad u(k) \in U \tag{7.2}$$

for all $k \geq 0$, where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are closed sets.

The steady state and input of the plant (x_s, u_s) are such that (7.1) is fulfilled, i.e. $x_s = f(x_s, u_s)$.
We define the set of admissible equilibrium states as

$$\mathcal{Z}_s = \{(x,u) \in X \times U \mid x = f(x,u)\} \tag{7.3}$$

$$\mathcal{X}_s = \{x \in X \mid \exists u \in U \text{ such that } (x,u) \in Z_s\} \tag{7.4}$$

Notice that $\mathcal{X}_s$ is the projection of $\mathcal{Z}_s$ onto X.

Assume that there existas a parameter $\theta \in \mathbb{R}^{n_\theta}$ such that the couple (x_s, u_s) is univocally defined:

$$x_s = g_x(\theta),\ u_s = g_u(\theta) \tag{7.5}$$

Define, then, $\Theta = \{\theta \in \mathbb{R}^{n_\theta} \mid (g_x(\theta), g_u(\theta)) \in \lambda Z_s\}$.

The controller design problem consists of deriving a control law that minimizes a given performance cost index

$$\sum_{j=0}^{N-1} l(x(j), u(j))$$

where $l(x, u)$ defines the economic stage cost.

The model is assumed to satisfy the following assumption:

Assumption 7.1

1. *Function $l(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a non-negative function for any (x, u).*
2. *The model function $f(x, u)$ and the economic stage cost function $l(x, u)$ are Lipschitz continuous in (x, u); that is there exist Lipschitz constants L_f, $L_l > 0$ such that, for all $(x, u), (x_0, u_0) \in X \times U$*

$$\begin{aligned} |f(x,u) - f(x_0,u_0)| &\leq L_f|(x,u) - (x_0,u_0)| \\ |l(x,u) - l(x_0,u_0)| &\leq L_l|(x,u) - (x_0,u_0)| \end{aligned}$$

3. (Weak controllability) *Define the set*

$$Z_N(w) = \{(x, \mathbf{u}) \in X \times U^N \mid x(j) \in X,\ u(j) \in U,\ j \in \mathbb{I}_{0:N-1},\ x(N) = w\}$$

for any $w \in \mathcal{X}_s$, where $x(j) = \phi(j; x, \mathbf{u})$. It is assumed that there exist a set $\Omega \subset \mathbb{R}^n$ and a $\mathcal{K}_\infty$-function η such that, for any $(w, v) \in Z_s$ and for any $(x, \mathbf{u})$ such that $(x - w) \in \Omega$ and $(x, \mathbf{u}) \in Z_N(w)$, then

$$\sum_{j=0}^{N-1} |u(j) - v| \leq \eta(|x - w|)$$

4. *The set of parameter Θ is a convex set.*

Remark 7.2 *If the set* $\Theta = \{\theta \in \mathbb{R}^{n_\theta} \mid (g_x(\theta), g_u(\theta)) \in \lambda Z_s\}$ *results to be non-convex, then a suitable convex set* Θ *contained in* $\{\theta \in \mathbb{R}^{n_\theta} \mid (g_x(\theta), g_u(\theta)) \in \lambda Z_s\}$ *must be chosen.*

Given the economic stage cost, the economic controller should steer the system to the optimal reachable steady state, which is defined as follows:

Definition 7.3 *The optimal reachable steady state and input,* (x_s^*, u_s^*)*, satisfy*

$$\begin{aligned}(x_s^*, u_s^*) \quad &= \quad \arg\min_{x,u} l(x,u) \\ s.t. \quad & \quad x = f(x,u) \\ & \quad x \in X, \quad u \in U\end{aligned}$$

7.3 Economic MPC

In the economic MPC the stage cost is an arbitrary economic objective, which does not necessarily penalizes the tracking error to the optimal target (x_s^*, u_s^*). In (Rawlings et al., 2008) and (Rawligns and Amrit, 2009), for instance, the authors use a stage cost of the optimal control problem which measures distance from the setpoint even if this setpoint is unreachable at steady state due to the problem constraints, but desirable for economic reasons. In general, the economic MPC cost function is given by

$$V_N^e(x, \mathbf{u}) = \sum_{j=0}^{N-1} l(x(j), u(j))$$

The economic MPC control law is derived from the solution of the optimization problem $P_N^e(x)$

$$\begin{aligned}&\min_{\mathbf{u}} V_N^e(x, \mathbf{u}) \\ &\quad s.t. \\ &\qquad x(0) = x, \\ &\qquad x^+ = f(x,u), \\ &\qquad x(j) \in X,\, u(j) \in U, \quad j \in \mathbb{I}_{0:N-1} \\ &\qquad x(N) = x_s^*\end{aligned}$$

and is given by the receding horizon application of the optimal solution, $\kappa_N^e(x) = u^0(0;x)$. The optimal value of the cost function is noted as $V_N^{e0}(x)$. The feasible region of the optimization problem is given by:

$$\mathcal{X}_N^e = \{x \in X \mid \exists (x, \mathbf{u}) \in \mathcal{Z}_N(w), \text{ for } w = x_s^*\}$$

The standard Lyapunov arguments to prove asymptotic stability of MPC cannot be used in this case because the optimal cost is not necessarily decreasing along the closed-loop trajectory. In (Rawlings et al., 2008) asymptotic stability for linear, stabilizable models with strictly convex quadratic cost function is established, but a Lyapunov function is not found. In the paper (Diehl et al., 2011), asymptotic stability of economic MPC is established using Lyapunov arguments. In order to find a suitable Lyapunov function, in (Diehl et al., 2011) the authors made the following assumption:

Assumption 7.4 (Strong duality of the steady state problem) *Let $L_r(x,u)$ be the* rotated *stage cost function given by*

$$L_r(x,u) = l(x,u) + \lambda^{'}(x - f(x,u)) - l(x_s, u_s)$$

where λ is a multiplier that ensures the rotated cost exhibits a unique minimum at (x_s, u_s) for all $x \in X$, $u \in U$. Then there exists a a $\mathcal{K}$-function α_1 such that $L_r(x,u) \geq \alpha_1(|x - x_s|)$.

In (Diehl et al., 2011) it is proved that the predictive control law derived from the following optimization problem $\tilde{P}_N^e(x)$

$$\begin{aligned}
&\min_{\mathbf{u}} \sum_{j=0}^{N-1} L_r(x(j), u(j)) \\
&s.t. \\
&\quad x(0) = x, \\
&\quad x^+ = f(x,u), \\
&\quad x(j) \in X,\ u(j) \in U, \quad j \in \mathbb{I}_{0:N-1} \\
&\quad x(N) = x_s^*
\end{aligned}$$

is identical to the economic predictive control law, and that the optimal cost function is a Lyapunov function, which demonstrates asymptotic stability as in standard MPC (Diehl et al., 2011).

Remark 7.5 (Convex problems) *In (Diehl et al., 2011) it is pointed out that in the convex case it is easy to show that, if the steady state problem is feasible and $l(.,.)$ is strictly convex in (x, u), Assumption 7.4 is always satisfied.*

When the economic objective of the controller changes, the optimal admissible steady state (x_s, u_s) (to which the system should be steered by the controller) changes as well. This change may cause a loss of feasibility of the controller.

At the same time, the MPC for tracking constrained linear and nonlinear systems presented in the previous chapters of this thesis, provides stability and convergence to the setpoint under changing operating points, but based on what has been explained in this section, it is suboptimal with respect to an economic objective.

The main objective of this chapter is to combine the economic MPC proposed in (Diehl et al., 2011) with the MPC for tracking proposed in (Limon et al., 2009b) in such a way that the combined controller inherits the advantages of both formulations.

7.4 Economic MPC for a changing economic criterion

In this section the economic MPC for a changing economic criterion is presented. The cost function proposed is composed by an economic stage cost function and an offset cost function as in (Limon et al., 2008a; Ferramosca et al., 2009a; Limon et al., 2009b). This offset cost function is defined as follows:

Definition 7.6 *Let $V_O(x, u)$ be a convex positive definite function such that the unique minimizer of*

$$\min_{(x,u)\in\mathcal{Z}_s} V_O(x, u)$$

is (x_s, u_s).

The economic stage cost function is given by:

$$\ell_t(z, v) = \ell(z + x_s^*, v + u_s^*)$$

To the aim of proving stability of the controller, we need to introduce the *rotated* stage cost function:

Definition 7.7

$$L_t(z, v) = L_r(z + x_s^*, v + u_s^*)$$

This stage cost function satisfies the following properties:

Property 7.8

1. $L_t(x - x_s^*, u - u_s^*) = L_r(x, u)$
2. $L_t(0, 0) = L_r(x_s^*, u_s^*) = 0$
3. $L_t(z, v) \geq \alpha_1(|z|) + \alpha_2(|v|)$ *for certain* $\mathcal{K}$ *functions* α_1 *and* α_2.

The economic MPC for a changing economic criterion proposed in this chapter solves, for any current state x, the following optimization problem $P_N(x)$:

$$\begin{aligned}
&\min_{\mathbf{u},\theta} V_N(x; \mathbf{u}, \theta) && (7.6)\\
&s.t. && (7.7)\\
&\quad x(0) = x, && (7.8)\\
&\quad x^+ = f(x, u), && (7.9)\\
&\quad x(j) \in X,\ u(j) \in U,\ j \in \mathbb{I}_{0:N-1} && (7.10)\\
&\quad x_s = g_x(\theta),\ u_s = g_u(\theta) && (7.11)\\
&\quad \theta \in \Theta && (7.12)\\
&\quad x(N) = x_s && (7.13)\\
& && (7.14)
\end{aligned}$$

where

$$V_N(x; \mathbf{u}, \theta) \quad = \quad \sum_{j=0}^{N-1} L_t(x(j) - x_s, u(j) - u_s) + V_O(x_s, u_s)$$

The optimal cost and the optimal decision variables will be denoted as $V_N^0(x)$ and $(\mathbf{u}^0, \theta^0)$ respectively. The control law is given by $\kappa_N(x) = u^0(0; x)$.

Notice that, if we add the constraint $x(N) = x_s^*$ in $P_N(x)$, the optimization problem is the same as $P_N^e(x)$.
The feasible region of the optimization problem is a compact set given by

$$\mathcal{X}_N = \{x \in X \mid \exists(x, \mathbf{u}) \in \mathcal{Z}_N(w),\ \text{for } w \in \mathcal{X}_s\}$$

Since $\{x_s^*\} \subset \mathcal{X}_s$, we have that $\mathcal{X}_N^e \subset \mathcal{X}_N$.
The set $\mathcal{X}_N$ is a feasible set of initial x such that one can reach *any* feasible steady state with N admissible inputs.
The set $\mathcal{X}_N^e$ is a feasible set of initial x such that one can reach *the optimal* steady state with N admissible inputs.
Then, the set $\mathcal{X}_N$ is larger, and, in some applications much larger, than $\mathcal{X}_N^e$, (Ferramosca et al., 2010b).

In the following theorem, asymptotic stability of the proposed economic MPC for tracking is stated.

Theorem 7.9 *If assumptions 7.1 and 7.4 hold, then (x_s^*, u_s^*) is an asymptotically stable equilibrium point for the controlled system and its domain of attraction is $\mathcal{X}_N$.*

The proof of Theorem 7.9 follows the same arguments as the proof of Theorem 1 in chapter 6

Proof: Consider that $x \in \mathcal{X}_N$ at time k, then the optimal cost function is given by $V_N^0(x) = V_N(x; \mathbf{u}^0(x), \theta^0(x))$, where $(\mathbf{u}^0(x), \theta^0(x))$ defines the optimal solution of $P_N(x, y_t)$ and $\mathbf{u}^0(x) = \{u^0(0;x), u^0(1;x), ..., u^0(N-1;x)\}$. Notice that $u^0(0;x) = \kappa_N(x)$. The resultant optimal state sequence associated to $\mathbf{u}^0(x)$ is given by $\mathbf{x}^0(x) = \{x^0(0;x), x^0(1;x), ..., x^0(N-1;x), x^0(N;x)\}$, where $x^0(0;x) = x$, $x^0(1;x) = x^+$ and $x^0(N;x) = x_s^0(x) = g_x(\theta^0(x))$.

As standard in MPC (Mayne et al., 2000; Rawlings and Mayne, 2009, Chapter 2), define the successor state at time $k+1$, $x^+ = f(x, \kappa_N(x))$ and define also the following sequences:

$$\begin{aligned} \tilde{\mathbf{u}} &\triangleq [u^0(1;x), \cdots, u^0(N-1;x), u_s^0(x)] \\ \tilde{\theta} &\triangleq \theta^0(x) \end{aligned}$$

where $u_s^0(x) = g_u(\theta^0(x))$. It is easy to derive that $(\tilde{\mathbf{u}}, \tilde{\theta})$ is a feasible solution for the optimization problem $P_N(x^+)$. Therefore, $\mathcal{X}_N$ is an admissible positive invariant set for the closed-loop system and hence the control law is well-defined and the constraints are fulfilled throughout the system evolution.

The state sequence due to $(\tilde{\mathbf{u}}, \tilde{\theta})$ is $\tilde{\mathbf{x}} = \{x^0(1;x), x^0(2;x), ..., x^0(N;x), x^0(N+1;x)\}$, where $x^0(N;x) = x_s^0(x)$ and $x^0(N+1;x) = f(x^0(N;x), u_s^0(x)) = x_s^0(x)$. Hence,

$$\tilde{\mathbf{x}} = \{x^0(1;x), x^0(2;x), ..., x_s^0(x), x_s^0(x)\}$$

which is clearly feasible. Compare now the optimal cost $V_N^0(x)$, with the cost given by $(\tilde{\mathbf{u}}, \tilde{\theta})$, $\tilde{V}_N(x^+; \tilde{\mathbf{u}}, \tilde{\theta})$. Taking into account the properties of the feasible nominal trajectories for x^+, Assumption 7.1 and using standard procedures in MPC (Mayne et al., 2000; Rawlings and Mayne, 2009, Chapter 2) it is possible to obtain:

$$\begin{aligned}\tilde{V}_N(x^+; \tilde{\mathbf{u}}, \tilde{\theta}) - V_N^0(x) &= -L_t((x - x_s^0(x)), (u^0(0;x) - u_s^0(x))) - V_O(x_s^0, u_s^0)\\ &\quad + L_t((x(N;x) - x_s^0(x)), (u_s^0(x) - u_s^0(x))) + V_O(x_s^0, u_s^0)\\ &= -L_t((x - x_s^0(x)), (u^0(0;x) - u_s^0(x)))\end{aligned}$$

By optimality, we have that $V_N^0(x^+) \leq \tilde{V}_N(x^+; \tilde{\mathbf{u}}, \tilde{\theta})$ and then:

$$\begin{aligned}V_N^0(x^+) - V_N^0(x) &\leq -L_t((x - x_s^0(x)), (u^0(0;x) - u_s^0(x)))\\ &= -L_t((x - x_s^0(x)), (\kappa_N(x) - u_s^0(x)))\end{aligned}$$

Taking into account the property 7.8, we have that:

$$\lim_{k\to\infty} |x(k) - x_s^0(x(k))| = 0, \quad \lim_{k\to\infty} |u(k) - u_s^0(x(k))| = 0$$

Hence the system converges to a point (x_s^0, u_s^0), such that $x_s^0 = g_x(\theta_s^0)$ and $u_s^0 = g_u(\theta_s^0)$.

Now, it is proved that the system converges to an equilibrium point. Pick an $\varepsilon > 0$, then there exists a $k(\varepsilon)$ such that for all $k \geq k(\varepsilon)$, $|x(k) - x_s^0(x(k))| < \varepsilon$ and $|u(k) - u_s^0(x(k))| < \varepsilon$. Then, removing the time dependence for the sake of simplicity, it is inferred that

$$\begin{aligned}|x^+ - x| &= |x^+ - x_s^0(x) + x_s^0(x) - x|\\ &\leq |x^+ - x_s^0(x)| + |x_s^0(x) - x|\\ &= |f(x,u) - f(x_s^0(x), u_s^0(x))| + |x_s^0(x) - x|\\ &\leq L_f|x - x_s^0(x)| + L_f|u - u_s^0(x)| + |x_s^0(x) - x|\\ &\leq (2L_f + 1)\varepsilon\end{aligned}$$

Therefore, for a given $\varepsilon > 0$, there exists a $k(\varepsilon)$ such that $|x^+ - x| \leq (2L_f + 1)\varepsilon$. Hence, the system converges to a steady state x_∞ and this is such that $x_\infty = x_s^0(x_\infty) \in \mathcal{X}_s$.

Since $V_O(x_s, u_s)$ is convex, using lemma 7.17, it is proved that (x_∞, u_∞) is the optimal steady state of the system, that is $(x_\infty, u_\infty) = (x_s^*, u_s^*)$.

Finally, the fact that (x_s^*, u_s^*) is a stable equilibrium point for the closed-loop system is proved. That is, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $|x(0) - x_s^*| \leq \delta$, then $|x(k) - x_s^*| \leq \varepsilon$.

To this aim, define the function $W(x) = V_N^0(x) - V_O(x_s^*, u_s^*)$. Then, $W(x_s^*) = 0$. This function is such that $\alpha_W(|x - x_s^*|) \leq W(x) \leq \beta_W(|x - x_s^*|)$, where α_W and β_W are suitable $\mathcal{K}_\infty$ functions. In fact:

- $W(x) \geq \alpha_l(|x - x_s^0|) + \alpha_O(|x_s^0 - x_s^*|)$. This comes from the fact that the stage cost function is a positive definite function and from the definition of V_O. Hence

$$\begin{aligned} W(x) &\geq \alpha_W(|x - x_s^0| + |x_s^0 - x_s^*|) \\ &\geq \alpha_W(|x - x_s^*|) \end{aligned}$$

- Notice that $V_N^0(x) \leq V_N(x; \mathbf{u}, \theta^*) + V_O(x_s^*, u_s^*)$. Due to the weak controllability of x_s^* (assumption 7.1), there exists a $\mathcal{K}_\infty$ function β_W such that $V_N(x; \mathbf{u}, \theta^*) \leq \beta_W(|x - x_s^*|)$. Hence $W(x) \leq \beta_W(|x - x_s^*|)$.

Then, $\alpha_W(|x(k) - x_s^*|) \leq W(x(k)) \leq W(x(0)) \leq \beta_W(|x - x_s^*|)$ and, hence, $|x(k) - x_s^*| \leq \alpha_W^{-1} \circ \beta_W(|x(0) - x_s^*|)$. So, picking $\delta = \beta_W^{-1} \circ \alpha_W(\varepsilon)$, then $|x(k) - x_s^*| \leq \alpha_W^{-1} \circ \beta_W(\delta) \leq \varepsilon$, proving the stability of x_s^*.

Recapping, it has been proved that for all initial state $x_0 \in \mathcal{X}_N$, the closed-loop system converges to an equilibrium point (x_s^*, u_s^*). Moreover, it has been demonstrated that this equilibrium point is stable for the closed-loop system. Therefore, (x_s^*, u_s^*) is an asymptotic stable equilibrium point for the closed-loop system and its domain of attraction is $\mathcal{X}_N$. ■

Property 7.10 (Changing economic criterion) *Since the set of constraints of $P_N(x)$ does not depend on (x_s^*, u_s^*), the proposed controller is able to guarantee the recursive feasibility, admissibility and asymptotic stability for any change on-line of the economic criterion. In fact, since the domain of attraction $\mathcal{X}_N$ does not depend on the optimal steady state, for all $x(0) \in \mathcal{X}_N$ every admissible steady state is reachable. Moreover, since the trajectory remains in $\mathcal{X}_N$, if the economic criterion (and hence the optimal steady state) changes, problem $P_N(x)$ does not loose feasibility and the system is led to the new optimal steady state in an admissible way.*

The drawback of this formulation is that it requires the *a priori* calculation of λ and (x_s^*, u_s^*). In the next section it is shown that in case of linear systems, the dependence on λ can be removed as in (Diehl et al., 2011), by means of a suitable choice of the offset cost function.

7.4.1 Specialization to the linear case

Consider that the model function $f(x, u)$ is linear, such that:

$$x^+ = Ax + Bu$$

Consider also that the pair (A,B) is stabilizable.

In order to prove that the dependence from λ can be removed, a rotated offset cost function can be defined:

Definition 7.11

$$\tilde{V}_O(x,u) \quad = \quad V_O(x,u) + \lambda'(x - x_s^*) - V_O(x_s^*, u_s^*)$$

Notice that this rotated offset cost function has to fulfill assumption 7.1.4. Hence, if $V_O(x, u)$ is convex, since $\lambda'(x - x_s^*)$ is convex, then $\tilde{V}_O(x,u)$ is also convex.

Hence, the cost function of the economic MPC problem for a changing economic criterion in case of linear systems is given by:

$$\tilde{V}_N(x; \mathbf{u}, \theta) = \sum_{j=0}^{N-1} L_t(x(j) - x_s, u(j) - u_s) + \tilde{V}_O(x_s, u_s)$$

and the optimization problem $\tilde{P}_N(x)$ is given by

$$\min_{\mathbf{u},\theta} \tilde{V}_N(x;\mathbf{u},\theta) \tag{7.15}$$

$$s.t. \tag{7.16}$$

$$x(0) = x, \tag{7.17}$$

$$x^+ = Ax + Bu, \tag{7.18}$$

$$x(j) \in X,\ u(j) \in U,\ \ j \in \mathbb{I}_{0:N-1} \tag{7.19}$$

$$x_s = g_x(\theta) \in \lambda X \tag{7.20}$$

$$u_s = g_u(\theta) \in \lambda U \tag{7.21}$$

$$x(N) = x_s \tag{7.22}$$

$$\tag{7.23}$$

Lemma 7.12 *The optimization problem $\tilde{P}_N(x)$ is equivalent to*

$$\min_{\mathbf{u},\theta} \sum_{j=0}^{N-1} \ell_t(x(j) - x_s, u(j) - u_s) + V_O(x_s, u_s) \tag{7.24}$$

$$s.t. \tag{7.25}$$

$$x(0) = x, \tag{7.26}$$

$$x^+ = Ax + Bu, \tag{7.27}$$

$$x(j) \in X,\ u(j) \in U,\ \ j \in \mathbb{I}_{0:N-1} \tag{7.28}$$

$$x_s = g_x(\theta) \in \lambda X \tag{7.29}$$

$$u_s = g_u(\theta) \in \lambda U \tag{7.30}$$

$$x(N) = x_s \tag{7.31}$$

$$\tag{7.32}$$

Proof: Operating with the cost function we have that

$$\begin{aligned}
\tilde{V}_N(x;\mathbf{u},\theta) &= \sum_{j=0}^{N-1} L_t(x(j)-x_s,u(j)-u_s)+\tilde{V}_O(x_s,u_s)\\
&= \sum_{j=0}^{N-1} \big(\ell_t(x(j)-x_s,u(j)-u_s)+\lambda^{'}(x(j)-x(j+1))-\ell_t(x_s^*,u_s^*)\big)\\
&\quad +V_O(x_s,u_s)+\lambda^{'}(x_s-x_s^*)-V_O(x_s^*,u_s^*)\\
&= \sum_{j=0}^{N-1} \ell_t(x(j)-x_s,u(j)-u_s)+\lambda^{'}(x-x_s)-N\ell_t(x_s^*,u_s^*)\\
&\quad +V_O(x_s,u_s)+\lambda^{'}(x_s-x_s^*)-V_O(x_s^*,u_s^*)\\
&= \sum_{j=0}^{N-1} \ell_t(x(j)-x_s,u(j)-u_s)+\lambda^{'}(x-x_s)-N\ell_t(x_s^*,u_s^*)\\
&\quad +V_O(x_s,u_s)-V_O(x_s^*,u_s^*)
\end{aligned}$$

This is equivalent to optimize, at any instant, the cost function

$$V_N(x;\mathbf{u},\theta) = \sum_{j=0}^{N-1} \ell_t(x(j)-x_s,u(j)-u_s)+V_O(x_s,u_s)$$

proving the lemma. ■

From this lemma it is clear that, solving problem (7.24) instead of problem (7.15), gives the same controller. The advantage is that is not necessary to know the value of λ.

7.5 Local economic optimality

The proposed economic MPC for a changing economic criterion may be considered as a suboptimal controller (with respect to the setpoint) due to the stage cost to minimize. As demonstrated in the following property, however, under mild conditions on the offset cost function $V_O(\cdot)$, the proposed controller ensures the economic optimality property as in (Diehl et al., 2011).

Assumption 7.13 *There exist a positive constant γ such that*

$$V_O(x,u) \geq \gamma|x-x_s^*|$$

Then we can state the following property:

Property 7.14 (Local optimality)
Consider that assumptions 7.1, 7.4 and 7.13 hold and assume that $x_0 \in \mathcal{X}_N$. *Then there exists a* $\alpha^0 > 0$ *such that for all* $\gamma \geq \alpha^0$ *and for all* $x \in \mathcal{X}_N^e$ *the proposed economic MPC for tracking is equal to the economic MPC, i.e.* $\kappa_N(x) = \kappa_N^e(x)$.

Proof: First, define problem $\hat{P}_N^e(x)$, which is equivalent to problem $P_N^e(x)$, but rewritten as follows:

$$\min_{\mathbf{u}} \sum_{j=0}^{N-1} L_t(x(j) - x_s, u(j) - u_s) + V_O(x_s, u_s) \tag{7.33}$$

$$s.t. \tag{7.34}$$

$$x(0) = x, \tag{7.35}$$

$$x^+ = f(x, u), \tag{7.36}$$

$$x(j) \in X,\ u(j) \in U,\ j \in \mathbb{I}_{0:N-1} \tag{7.37}$$

$$x_s = g_x(\theta) \in \lambda X \tag{7.38}$$

$$u_s = g_u(\theta) \in \lambda U \tag{7.39}$$

$$x(N) = x_s \tag{7.40}$$

$$|x(N) - x_s^*|_q = 0 \tag{7.41}$$

Let $\nu(x)$ be the Lagrange multiplier of the equality constraint $|x(N) - x_s^*|_q = 0$ of the optimization problem $\hat{P}_N^e(x)$. We define the following constant α^0

$$\alpha^0 \;=\; \max_{x \in \mathcal{X}_N^e} |\nu(x)|$$

Define the optimization problem $\tilde{P}_{N,\gamma}(x)$ as a particular case of $P_N(x)$ with $V_O(x, u) \triangleq \gamma |x - x_s^*|_p$, where $|.|_p$ is the dual norm of $|.|_q$[1]. This optimization problem results from $\hat{P}_N^e(x)$ with the last constraint posed as an exact penalty function. Therefore, in virtue of the well-known result on the exact penalty functions (Luenberger, 1984), taking any $\gamma \geq \alpha^0$ we have that

$$V_N^{e0}(x) = \tilde{V}_{N,\alpha^0}^0(x) \leq \tilde{V}_{N,\gamma}^0(x) \leq V_N^{e0}(x)$$

and hence $\tilde{V}_{N,\gamma}^0(x) = V_N^{e0}(x)$ for all $x \in \mathcal{X}_N^e$. ■

[1] The dual $|.|_p$ of a given norm $|.|_q$ is defined as $|u|_p \triangleq \max_{|v|_q \leq 1} u'v$. For instance, $p = 1$ if $q = \infty$ and vice versa, or $p = 2$ if $q = 2$ (Luenberger, 1984).

7.5.1 Specialization to the linear case

In case of linear systems, we need to make the following assumption, due to the definition of the rotated offset cost function 7.11

Assumption 7.15 *There exist a positive constant $\hat{\gamma}$ such that*

$$V_O(x,u) - V_O(x_s^*, u_s^*) \geq \hat{\gamma}|x - x_s^*|$$

Lemma 7.16
If assumption 7.15 holds, then for any $\gamma > 0$ there exists a $\hat{\gamma} > 0$ such that

$$\tilde{V}_O(x,u) \geq \gamma|x - x_s^*|$$

Proof: Using the Cauchy−Schwarz's inequality, $\lambda'(x_s - x_s^*) \leq |\lambda||x_s - x_s^*|$. Take $\hat{\gamma} > |\lambda| + \gamma$. Then,

$$\begin{aligned}
\tilde{V}_O(x_s, u_s) &= V_O(x_s, u_s) + \lambda'(x_s - x_s^*) - V_O(x_s^*, u_s^*) \\
&\geq \hat{\gamma}|x_s - x_s^*| + \lambda'(x_s - x_s^*) \\
&\geq |\lambda||x_s - x_s^*| + \gamma|x_s - x_s^*| + \lambda'(x_s - x_s^*) \\
&= [|\lambda||x_s^* - x_s| - \lambda'(x_s^* - x_s)] + \gamma|x_s - x_s^*| \\
&\geq \gamma|x_s - x_s^*|
\end{aligned}$$

■

7.6 Illustrative examples

In this section three examples are presented. The first one shows that the economic MPC for a changing economic criterion inherits the large feasible set associated with MPC for tracking (Limon et al., 2008a; Ferramosca et al., 2009a). In the second example, the role of the offset cost function in the local economic optimality property is shown. The third example shows that the economic MPC for a changing economic criterion inherits the economic optimality property of the economic MPC (Rawlings et al., 2008; Diehl et al., 2011).

7.6.1 Feasibility: the CSTR case

The system considered is a continuous stirred tank reactor (CSTR), (Chisci et al., 2005; Magni et al., 2001a). Assuming constant liquid volume, the CSTR for an exothermic, irreversible reaction, $A \to B$, is described by the following model:

$$\begin{aligned}\dot{C}_A &= \frac{q}{V}(C_{Af} - C_A) - k_o e^{(\frac{-E}{RT})} C_A \\ \dot{T} &= \frac{q}{V}(T_f - T) - \frac{\Delta H}{\rho C_p} k_o e^{(\frac{-E}{RT})} C_A + \frac{UA}{V\rho C_p}(T_c - T)\end{aligned} \tag{7.42}$$

where C_A is the concentration of A in the reactor, T is the reactor temperature and T_c is the temperature of the coolant stream. The nominal operating conditions are: $q = 100$ l/min, $T_f = 350$ K, $V = 100$ l, $\rho = 1000$ g/l, $C_p = 0.239$ J/g K, $\Delta H = -5 \times 10^4$ J/mol, $E/R = 8750$ K, $k_0 = 7.2 \times 10^{10}\text{min}^{-1}$, $UA = 5 \times 10^4$ J/ min K and $C_{Af} = 1$ mol/l.

The objective is to regulate $y = x_2 = T$ and $x_1 = C_A$ by manipulating $u = T_c$. The constraints are $0 \leq C_A \leq 1$ mol/l, $280\text{K} \leq T \leq 370\text{K}$ and $280\text{K} \leq T_c \leq 370$ K. The nonlinear discrete time model of system (7.42) is obtained by discretizing equation (7.42) using a 5-th order Runge-Kutta method and taking as sampling time 0.03 min. The set of reachable output is given by $304.17\text{K} \leq T \leq 370\text{K}$. The outputs in this range are all controllable.

The economic stage cost function is $l(x, u) = \|x - x_{sp}\|_Q^2 + \|u - u_{sp}\|_R^2$ where (x_{sp}, u_{sp}) defines the unreachable setpoint and $Q = diag(1, 1/100)$ and $R = 1/100$ are the weighting matrices. The function $V_O = \alpha\|x_s - x_{sp}\|_\infty$ has been chosen as the offset cost function. The controller has been implemented in MATLAB 7.8 and the function `fmincon` has been used to solve the optimization problem.

In Figure 7.1 the the evolution of the system for a change of setpoint is plotted. The system has been considered to be steered from $x_0 = (0.7950, 332)$, $u_0 = 302.8986$, to $y_{sp} = 400$, and then to $y_{sp} = 300$. Both setpoints are unreachable. The optimal equilibrium points are $x_s^* = (0.2057, 370)$, and $x_s^* = (0.9774, 304.17)$. A horizon $N = 3$ has been used. The evolution of the system (solid line), the artificial reference[2] x_s (dashed line), and the real reference (dashed-dotted line) are shown. Notice that the controller steers the system to the extremes of the reachable range, even with the short horizon.

[2]As stated in chapter 2 of this thesis, the equilibrium point (x_s, u_s) can be understood as an artificial reference that the system could reach with a feasible evolution.

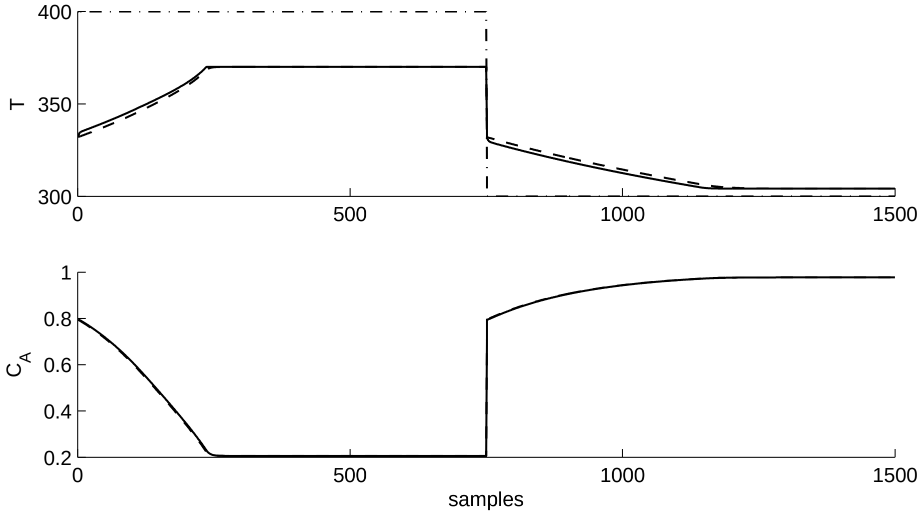


Figure 7.1: Time evolution of T and C_A.

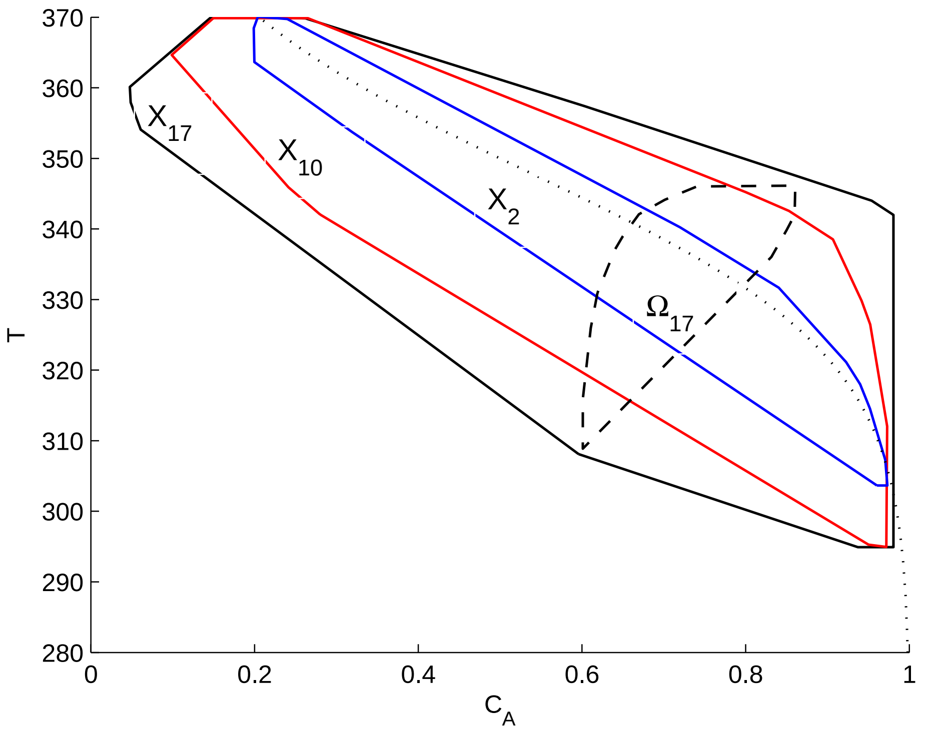


Figure 7.2: Feasible sets for different values of N.

In figure 7.2 the feasible sets $\mathcal{X}_N$ of the proposed controller for $N = 2$, $N = 10$ and

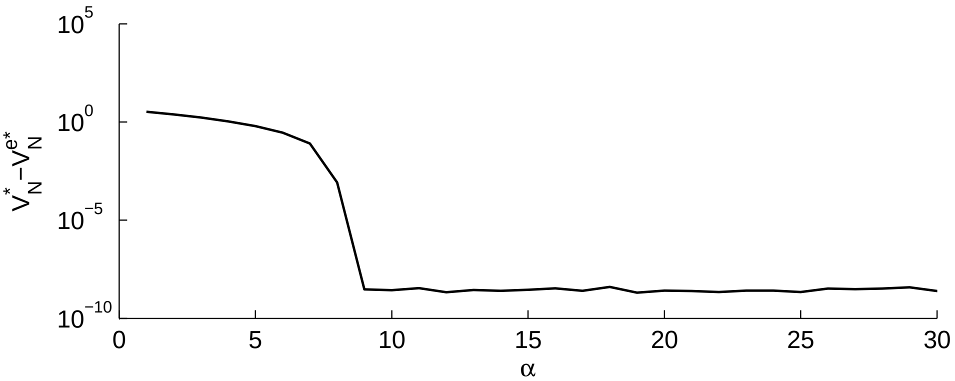


Figure 7.3: Difference between V_N^0 and V_N^{e0} versus α.

$N = 17$ are depicted in solid blue, red and black lines, respectively. These regions have been estimated solving a *Phase I* problem (Boyd and Vandenberghe, 2006) in a grid. The dotted line represents the set of equilibrium points of the system. The controller has been compared with the economic MPC presented in (Diehl et al., 2011). The resulting feasible set for $N = 17$, labeled as Ω_{17} is depicted in dashed line. Notice that this set is smaller than the one obtained with the economic MPC for a changing economic criterion using an horizon $N = 17$. Moreover, notice how the sets X_N cover the entire steady state manifold, even for $N = 2$, thus showing that the new formulation has increased the domain of attraction of the controller.

7.6.2 Local optimality

In this example, the local optimality property is shown. To this aim, the difference between the optimal cost of the MPC presented in (Diehl et al., 2011), V_N^{e0}, and the one presented in this chapter, V_N^0, are compared.

The test consisted in calculating the optimal costs V_N^0 and V_N^{e0}, for steering the system from $x_0 = (0.7950, 332)$, $u_0 = 302.8986$, to $x_{sp} = (1, 340)$, $u_{sp} = 350$, which is not a reachable equilibrium point for the system. The optimal equilibrium point for the system, given x_{sp} and u_{sp}, is $x_s^* = (0.7255, 336.9019)$, $u_s^* = 303.1932$. V_N^0, with $N = 17$, has been calculated for different values of α. In figure 7.3 the value of $V_N^0 - V_N^{e0}$ versus α is plotted. As can be seen, $V_N^0 - V_N^{e0}$ drops sharply to near zero for $\alpha = 9$. As discussed in section 7.5, this happens because the value of α becomes greater than the value of the Lagrange multiplier of the last equality constraint of problem P_N^{e0}, which is $\alpha^0 = 8.1127$.

7.6.3 Economic Optimality

The object of this example is to show that the controller proposed in this paper inherits the optimality with respect to the setpoint property from the economic MPC formulation (Rawlings et al., 2008; Diehl et al., 2011). To this aim, a comparison of the optimality performance of the three controllers - economic MPC, MPC for tracking (Limon et al., 2008a; Ferramosca et al., 2009a), economic MPC for a changing economic criterion - has been made.

The cost functions are the following:
Economic MPC (E-MPC, (Rawlings et al., 2008; Diehl et al., 2011)):

$$V_N^e(x;\mathbf{u}) \;=\; \sum_{j=0}^{N-1} \ell(x(j),u(j))$$

with the economic stage cost function considered as in the example presented in section 7.6.1, that is $\ell(x,u) = |x - x_{sp}|_Q^2 + |u - u_{sp}|_R^2$ where (x_{sp}, u_{sp}) defines the unreachable setpoint and Q and R are the weighting matrices.
MPC for tracking (MPCT, (Limon et al., 2008a; Ferramosca et al., 2009a)):

$$V_N^t(x;\mathbf{u},\theta) \;=\; \sum_{j=0}^{N-1} |x(j) - x_s|_Q^2 + |u(j) - u_s|_R^2 + |x(N) - x_{sp}|_T^2$$

where $|x(N) - x_{sp}|_T^2$ is the offset cost function.
Economic MPC for a changing economic criterion (E-MPCT):

$$V_N(x;\mathbf{u},\theta) \;=\; \sum_{j=0}^{N-1} \ell_t(x(j) - x_s, u(j) - u_s) + V_O(x_s,u_s)$$

The offset cost function considered for the economic MPC for changing economic criterion is $V_O = \alpha|x_s - x_{sp}|_\infty$, where α is chosen as the value of the Lagrange multiplier of the last equality constraint of problem P_N^{e0} (see section 7.5).

The controllers' performance have been assessed using the following closed-loop control performance measure:

$$\Phi \;=\; \sum_{k=0}^{T} |x(k) - x_{sp}|_Q^2 + |u(k) - u_{sp}|_R^2 - (|x_s^* - x_{sp}|_Q^2 + |u_s^* - u_{sp}|_R^2)$$

where T is the simulation time.

The system considered is the double integrator:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0 & 0.5 \\ 1.0 & 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hard constraints on state and input have been considered: $\|x\|_\infty \leq 5$ and $\|u\|_\infty \leq 0.3$.

The setpoint considered is $x_{sp} = (6; 3)$, which is unreachable. The optimal steady state, for the considered set of constraints, is $x_s^* = (5; 0.15)$. The MPC parameters are $Q = I_2$ and $R = I_2$. The simulation time is $T = 50$.

The initial condition considered is $x_0 = (-4; 2)$. A horizon $N = 10$ has been used. The value of α for the offset cost function has been chosen as $\alpha = 373$. As discussed in section 7.5, this value has been chosen greater than the value of the Lagrange multiplier of the last equality constraint of problem P_N^{e0}, which is $\alpha^0 = 372.64$.

The closed-loop performances of the three controllers are shown in table 7.1.

Table 7.1: Comparison of controller performance

Measure	E-MPC	MPCT	E-MPCT
Φ	226.7878	304.3342	226.7878

The performance of the new formulation is equal to the economic MPC performance, while the MPC for tracking has significantly worse performance. This shows how the new formulation inherits the optimality with respect to the setpoint from the economic MPC.

The closed-loop performances for a second simulation are shown in table 7.2. The setpoint considered has been $x_{sp} = (4.85; 3)$, which is unreachable. The optimal steady state is $x_s = (4.85; 0.15)$. The MPC parameters are $Q = I_2$ and $R = I_2$. The simulation time is $T = 50$. The initial condition considered is $x_0 = (0; -1.5)$. A horizon $N = 3$ has been used. With this shorter horizon, the initial state is infeasible for the economic MPC controller, but remains feasible for the other two controllers.

Notice that the new formulation again gives better closed-loop performance com-

Table 7.2: Comparison of controller performance

Measure	E-MPC	MPCT	E-MPCT
Φ	–	422.8128	385.1432

pared to MPC for tracking.

In Figure 7.4 the state space evolutions are depicted. The dashed line represents the evolution of the E-MPCT presented in the paper, while the dashed-dotted one represents the MPCT evolution. The feasible set for the proposed controller $\mathcal{X}_3$ (in solid edge) and the one for the economic MPC $\mathcal{X}_3^e$ (in dashed edge) are also depicted. The starting point x_0 is depicted as a dot, while the optimal steady state x_s is depicted as a star. It is clear how the two controllers have the same evolution while their trajectories lie outside of $\mathcal{X}_3^e$. This is because the E-MPCT follows the MPCT feasible trajectory while the E-MPC is unfeasible for a $N = 3$ horizon. When the trajectories enter the feasible set of the E-MPC, $\mathcal{X}_3^e$, the evolution of the system controlled with the E-MPCT follows the optimal one.

This example illustrates the two main properties of the economic MPC for tracking: it provides optimality with respect to the setpoint as in the economic MPC, and a large feasible regions as in the MPC for tracking formulation.

7.7 Conclusions

In this chapter, an MPC that handles a changing economic criterion has been presented, which is a hybrid of the MPC for setpoint tracking (Limon et al., 2008a; Ferramosca et al., 2009a) and the economic MPC (Rawlings et al., 2008; Diehl et al., 2011).
The results presented in this chapter have shown how the new formulation inherits the main properties of the two other controllers: the feasibility guaranty of the MPC for tracking and the optimality with respect to the setpoint of the economic MPC. The presented controller is able to provide a larger domain of attraction, as the MPC for tracking, and at the same time, a better performance with respect to the setpoint, as the economic MPC. Asymptotic stability of the proposed controller has been also established.

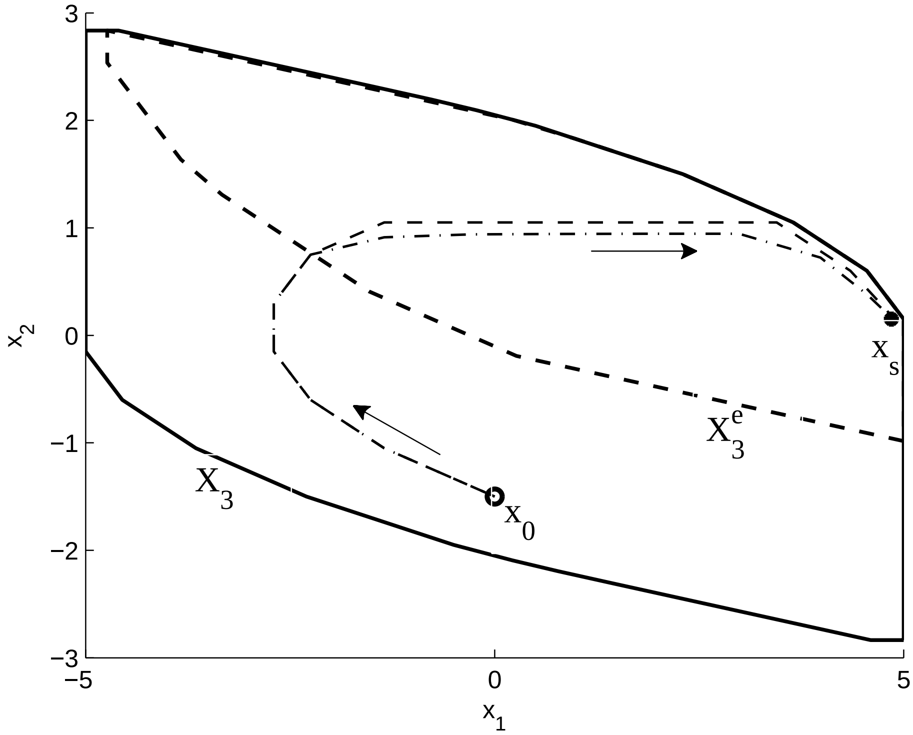


Figure 7.4: Feasible sets of the E-MPCT (solid edge) and of the E-MPC (dashed edge) and state evolutions of the MPCT (dashed-dotted line) and of the E-MPCT (dashed line). The E-MPCT follows the MPCT feasible trajectory while the E-MPC problem is unfeasible with $N = 3$ moves. When the trajectory enters $\mathcal{X}_3^e$, the E-MPCT follows the E-MPC optimal trajectory.

7.8 Appendix

In this Appendix section, the technical lemma used to prove Theorems 7.9 is presented. In particular, this lemmas prove the optimality of the steady state.

Lemma 7.17 *Consider system* (7.1) *subject to constraints* (7.2). *Consider that assumptions 7.1 and 7.4 hold. Assume that for a given state x the optimal solution of $P_N(x)$ is such that $x_s^0(x) = g_x(\theta^0(x))$ and $u_s^0(x) = g_u(\theta^0(x))$. Let $\tilde{\theta} \in \Theta$ be such that $\tilde{x}_s = g_x(\tilde{\theta})$ and $\tilde{u}_s = g_u(\tilde{\theta})$ are given by*

$$(\tilde{x}_s, \tilde{u}_s) \triangleq \arg \min_{(x_s,u_s)\in\mathcal{Z}_s} V_O(x_s, u_s)$$

Then

$$x_s^0(x) = \tilde{x}_s,\ u_s^0(x) = \tilde{u}_s$$

Proof: Consider that the optimal solution of $P_N(x)$ is θ^0. In the following, the dependence of the optimal solution from x will be omitted for the sake of clarity. The optimal cost function is $V_N^0(x) = V_O(x_s^0, u_s^0)$.

The lemma will be proved by contradiction. Assume that $\theta^0 \neq \tilde{\theta}$.

Define $\hat{\theta}$ given by

$$\hat{\theta} = \beta\theta^0 + (1-\beta)\tilde{\theta} \quad \beta \in [0,1]$$

Assuming Θ convex, hence $(\hat{x}_s, \hat{u}_s) = (g_x(\hat{\theta}), g_u(\hat{\theta}))$ is an admissible steady state. Therefore, defining as $\mathbf{u}$ the sequence of control actions derived from the control law $k(x, \hat{\theta})$, it is easily inferred that $(\mathbf{u}, \hat{\theta})$ is a feasible solution for $P_N(x_s^0)$. Then using standard procedures in MPC, we have that

$$\begin{aligned} V_N^0(x_s^0) &= V_O(x_s^0, u_s^0) \\ &\leq V_N(x_s^0; \mathbf{u}, \hat{\theta}) \\ &= \sum_{j=0}^{N-1} L_t((x(j) - \hat{x}_s), (k(x(j), \hat{y}_s) - \hat{u}_s)) + V_O(\hat{x}_s, \hat{u}_s) \\ &= V_O(\hat{x}_s, \hat{u}_s) \end{aligned}$$

Define $W(x_s^0, \beta) \triangleq V_O(V_O(\hat{x}_s, \hat{u}_s))$ and notice that $W(x_s^0, \beta) = V_N^0(x_s^0)$ for $\beta = 1$. Taking the partial of W about β we have that

$$\frac{\partial W}{\partial \beta} = g'(x_s^0 - \tilde{x}_s, u_s^0 - \tilde{u}_s)$$

where $g' \in \partial V_O(\hat{x}_s, \hat{u}_s)$, defining $\partial V_O(\hat{x}_s, \hat{u}_s)$ as the subdifferential of $V_O(\hat{x}_s, \hat{u}_s)$. Evaluating this partial for $\beta = 1$ we obtain that:

$$\left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} = g^{0'}(x_s^0 - \tilde{x}_s, u_s^0 - \tilde{u}_s)$$

where $g^{0'} \in \partial V_O(x_s^0, u_s^0)$, defining $\partial V_O(x_s^0, u_s^0)$ as the subdifferential of $V_O(x_s^0, u_s^0)$. Taking into account that V_O is a subdifferentiable function, from convexity (Boyd and Vandenberghe, 2006) we can state for every θ^0 and $\tilde{\theta}$ that

$$g^{0'}(x_s^0 - \tilde{x}_s, u_s^0 - \tilde{u}_s) \geq V_O(x_s^0, u_s^0) - V_O(\tilde{x}_s, \tilde{u}_s)$$

Taking into account that $\theta^0 \neq \tilde{\theta}$, $V_O(x_s^0, u_s^0) - V_O(\tilde{x}_s, \tilde{u}_s) > 0$, it can be derived that

$$\left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} \geq V_O(x_s^0, u_s^0) - V_O(\tilde{x}_s, \tilde{u}_s) > 0$$

This means that there exists a $\beta \in [\hat{\beta}, 1)$ such that $W(x_s^0, \beta)$ is smaller than the value of $W(x_s^0, \beta)$ for $\beta = 1$, which equals to $V_N^0(x_s^0)$.

This contradicts the optimality of the solution and hence the result is proved. ■

Chapter 8

Conclusions and future work

8.1 Contribution of the thesis

This thesis dealt with the problem of designing a model predictive controller (MPC) for process industries systems characterized by changes in their operating point. The traditional MPC formulation to regulate the system the a desired setpoint guarantees the setpoint tracking when there are no constraints but may not solve the problem when the plant has constraints. In this case, the change of setpoint may cause a loss of feasibility of the optimization problem, mainly because of two reasons: (i) the terminal set shifted to the new operating point may not be an admissible invariant set, which means that the feasibility property may be lost and (ii) the terminal region at the new setpoint could be unreachable in N steps, which makes the optimization problem unfeasible. In this case, a re-calculation of an appropriate value of the prediction horizon is necessary to ensure feasibility. Therefore, this would require an on-line re-design of the controller for each setpoint, which can be computationally unaffordable.

In Chapter 2, an MPC formulation able to overcome this problem has been presented. This formulation is characterized by the use of an artificial steady state considered as decision variable, the use of a cost function which measures the distance of the predicted trajectory to the artificial steady state, an additional cost that penalizes the distance of the artificial steady state to the desired output (the offset cost function), and an extended terminal constraint, the invariant set for tracking. It is proved in this chapter, that a suitable choice of the offset cost function ensures the local optimality property of the controller, that is, the MPC for tracking provides the same optimality as the unconstrained LQR. Moreover, it is presented a characterization of the region in which this property is ensured.
Besides, the proposed MPC for tracking formulation allows to consider any set of process variables as target which makes the controller suitable for non-square plants.

The proposed MPC for tracking deals with the case that the target to track does not fulfil the hard constraints or it is not an equilibrium point of the linear model. In this case the proposed controller steers the system to an admissible steady state (different to the target) which minimizes the offset cost function. This property means that the offset cost function plays the same role as a real-time optimizer (RTO), which is built in the proposed MPC. In Chapter 3 it has been also proved that, this function can be formulated as a distance to a set. This formulation makes the MPC for tracking suitable also for zone control problems, where the desired setpoint is not a fixed-point, but the output are desired to lie in a set. It has been proved in this chapter that the MPC for tracking target sets ensures recursive feasibility and stability. The issue of this formulation is the way the optimization problem can be solved, due to the non-trivial choice of the offset cost function. In the chapter, three formulations of the offset cost function, that allow to formulate the optimization problem as a QP problem, have been given.

In chapter 4, a robust MPC for tracking formulation for the zone control problem has been presented, for the case of presence of additive disturbances. The proposed controller is an MPC based on nominal predictions and constraints that get restricted each prediction step. It has been proved that this controller ensures stability, robust satisfaction of the constraints and recursive feasibility. The plant is assumed to be modeled as a linear system with additive uncertainties confined to a bounded known polyhedral set. It has been proved that, under mild assumptions, the proposed MPC is feasible under any change of the controlled variables target and steers the uncertain system to (a neighborhood of) the target if this is admissible. If the target is not admissible, the system is steered to the closest admissible operating point.

The thesis also focused on the problem of control of large scale systems. This kind of systems usually consist of linked unit of operations and can be divided into a number of subsystems, connected by networks of different nature. The overall control of these plants by means of a centralized controller is difficult to realize, because of the elevate computational burden and the difficult to manage the interchanges of information between the single units. Hence, an alternative control strategy is distributed control that is, a control strategy based on different agents - instead of a centralized controller - controlling each subsystems, which may or may not share information. The difference between these distributed control strategies is in the use of this open-loop information: noncooperative controllers, where each agent makes decision on the single subsystem considering the other subsystems information only locally and which make the plant converge to a Nash equilibrium; cooperative distributed controllers, which consider the effect of all the control actions on all subsystems in the network and make the system converging to the Pareto optimum. In Chapter 5 a cooperative distributed MPC for tracking linear systems is presented. In this formulation, each agent knows the overall plant objective, and optimizes it only with respect to its particular control action. As for the target problem, the controller has been implemented with a centralized offset cost function and a centralized terminal constraint. It has been proved that, the proposed controller ensures recursive feasibility and convergence to the centralized target.

The thesis has also dealt with nonlinear systems. In particular, in Chapter 6, the MPC for tracking has been extended to cope with nonlinear systems. Three formulations of this controller have been proposed, respectively characterized by equality terminal constraint, inequality terminal constraint and the absence of a terminal constraint. Stability and recursive feasibility have been proved for all these formulations. The calculation of the terminal ingredients, in the case of inequality terminal constraint, is not trivial. This chapter also proposed an interesting method for their calculation, based on an LTV modeling of the plant. The idea is to design a set of local predictive controllers, whose feasible regions cover the entire steady state manifold. The MPC for tracking *jumps* from a control law to another, according to the position of the artificial steady state in the steady state subspace.

Finally, the thesis focused on the topic of economic MPC. The economic MPC controllers are characterized by measuring an economic performance, instead of a distance to a setpoint as in the stantad tracking formulation. This economic performance can be measured by putting in the MPC cost function the real (possibly unreachable) target or by minimizing a generic cost according to some economic criterion. This economic MPC formulation has shown to provide better optimal performance with respect to the setpoint (or the economic criterion considered) than the standard tracking formulation. But when the economic criterion changes, the feasibility of the optimization problem may be lost. In Chapter 7, an economic MPC for a changing economic criterion has been presented. The main advantage of this controller is that it is able to provide the optimality properties of the economic MPC, and at the same time ensure feasibility under any change of the economic criterion, enlarging also the domain of attraction of the controller.

Summarizing, the aim of this thesis has been to study the problem of changing setpoint in system controlled by an MPC controller. The formulation proposed has the great advantage of always ensuring feasibility under any changes of the setpoint. The theoretical and simulated results provided, show that how this formulation can cope with linear and nonlinear systems, large scale system, zone control problems, economic problems, ensuring in any case local optimality, recursive feasibility and asymptotic stability.

8.2 Future work

In this section, some possible lines of research deriving from this thesis are presented.

- Formulation of robust MPC for tracking controllers. The presence of disturbances in real application is very common. Hence no control formulation can avoid this problem. The controllers presented in this thesis can be all robustified using the formulation presented in Chapter 4, or the well known tube-based formulation (Mayne et al., 2006; Limon et al., 2010a). Another interesting research direction would be the min-max formulation of the MPC for tracking.
- Formulation of the economic MPC for a changing economic criterion, with terminal inequality constraint. This formulation would allow to find a controller that provide an even larger domain of attraction. It would be also interesting to find a Lyapunov function, to prove asymptotic stability using Lyapunov arguments.
- Formulation of the MPC for tracking target set for large scale systems. The zone control problem are typical in process industries, as well as the presence of large scale system. A possible new direction of research would be the formulation of the distributed version of the MPC for tracking target sets.
- Different distributed formulation of the MPC for tracking. In this thesis the cooperative MPC formulation has been considered, for solving the distributed problem. An interesting research direction would be studying different MPC for tracking distributed formulations, like the noncooperative, the decentralized or the game-theory based one. Another interesting line would be also studying different ways of solving the target problem, instead of the centralized one.
- Application to real plants. The results presented in the thesis are mainly theoretical and have been tested only in simulation. The application of the proposed controllers to real plants, in order to verify the properties of the proposed controllers, would be a natural consequence of this thesis.

Appendix

Appendix A

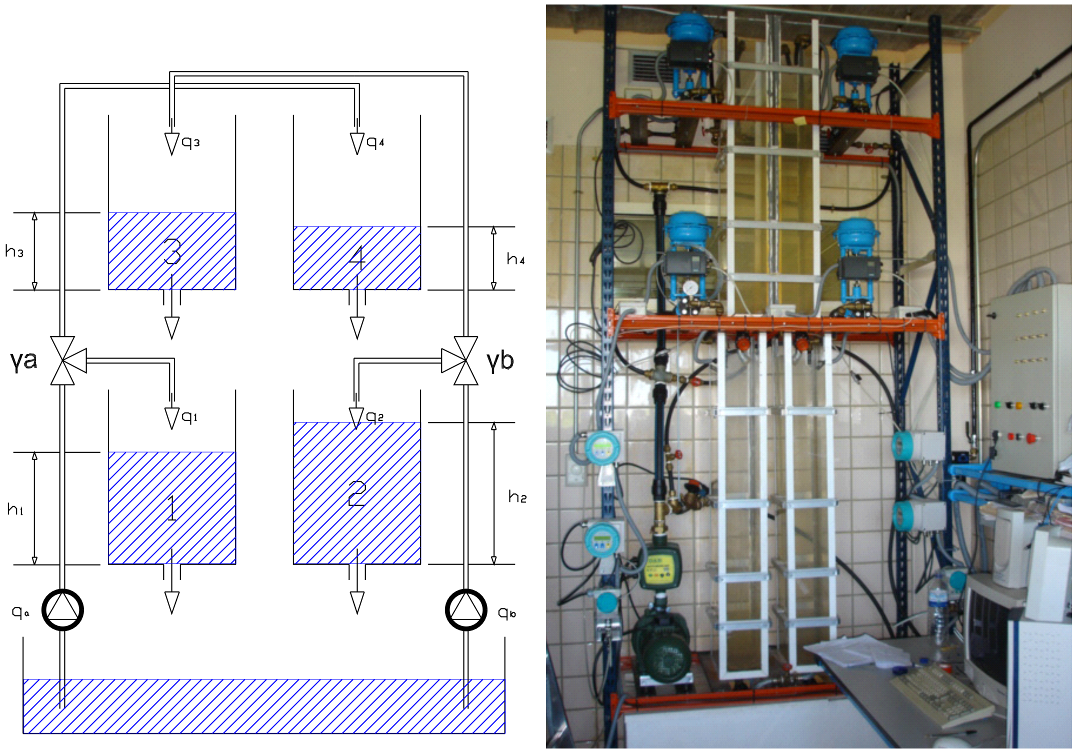
The four tanks process

In this appendix chapter, the system considered for the examples simulations of this thesis is introduced.

A.1 The 4 tanks process

The four tanks plant (Johansson, 2000) is a multivariable laboratory plant of interconnected tanks with nonlinear dynamics and subject to state and input constraints. One important property of this plant is that it can be configured to work at operation points characterized by multivariable zeros (minimum and non-minimum phase). A scheme of this plant is presented in Figure A.1(a). The inputs are the voltages of the two pumps and the outputs are the water levels in the lower two tanks.
A real experimental plant developed at the University of Seville is presented in Figure A.1(b). The real plant can be modified to offer a wide variety of configurations such as one single tank, two or three cascaded tanks, a mixture process and hybrid dynamics. Moreover the parameters that defined the dynamics of each tank can be modified by tuning the cross-section of the outlet hole of the tank. The real plant has been implemented using industrial instrumentation and a PLC for the low level control. Supervision and control of the plant is carried out in a computer by means of OPC (ole for process control) which allows one to connect the plant with a wide range of control programs such as LabView, Matlab or an industrial SCADA. Additional information on the quadruple tank process of the University of Seville are given in (Alvarado, 2007).

A state space continuous time model of the quadruple tank process system (Johansson, 2000) can be derived from first principles as follows



(a) Scheme of the 4 tank process.

(b) The real plant.

Figure A.1: The 4 tanks process.

$$
\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_a}{A_1}q_a \\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_b}{A_2}q_b \\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_b)}{A_3}q_b \\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_a)}{A_4}q_a
\end{aligned} \tag{A.1}
$$

The plant parameters, estimated on the real plant are shown in table A.1.

The minimum level of the tanks has been taken greater than zero to prevent eddy effects in the discharge of the tank. One important property of this plant is that the dynamics present multivariable transmission zeros which can be located in the right hand side of the s plane for some operating conditions. Hence, the values of γ_a and γ_b

have been chosen in order to obtain a system with non-minimum phase multivariable zeros.

Linearizing the model at an operating point given by h_i^0 and defining the deviation variables $x_i = h_i - h_i^o$ and $u_j = q_j - q_j^o$ where $j = a, b$ and $i = 1, \cdots, 4$ we have that:

$$
\begin{aligned}
\frac{dx}{dt} &= \begin{bmatrix} \dfrac{-1}{\tau_1} & 0 & \frac{A_3}{A_1\tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{A_4}{A_2\tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} \dfrac{\gamma_a}{A_1} & 0 \\ 0 & \frac{\gamma_b}{A_2} \\ 0 & \frac{(1-\gamma_b)}{A_3} \\ \frac{(1-\gamma_a)}{A_4} & 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x
\end{aligned}
$$

where $\tau_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}} \geq 0$, $i = 1, \cdots, 4$, are the time constants of each tank. This model has been discretized using the zero-order hold method with a sampling time of 5 seconds.

Table A.1: 4 tanks plant parameters

	Value	Unit	Description
H_{1max}	1.36	m	Maximum level of the tank 1
H_{2max}	1.36	m	Maximum level of the tank 2
H_{3max}	1.30	m	Maximum level of the tank 3
H_{4max}	1.30	m	Maximum level of the tank 4
H_{min}	0.2	m	Minimum level in all cases
Q_{amax}	3.6	m^3/h	Maximum flow of pump A
Q_{bmax}	4	m^3/h	Maximum flow of pump B
Q_{min}	0	m^3/h	Minimal flow
Q_a^0	1.63	m^3/h	Equilibrium flow (Q_a)
Q_b^0	2.0000	m^3/h	Equilibrium flow (Q_b)
a_1	1.310e-4	m^2	Discharge constant of tank 1
a_2	1.507e-4	m^2	Discharge constant of tank 2
a_3	9.267e-5	m^2	Discharge constant of tank 3
a_4	8.816e-5	m^2	Discharge constant of tank 4
A	0.06	m^2	Cross-section of all tanks
γ_a	0.3		Parameter of the 3-ways valve
γ_b	0.4		Parameter of the 3-ways valve
h_1^0	0.6487	m	Equilibrium level of tank 1
h_2^0	0.6639	m	Equilibrium level of tank 2
h_3^0	0.6498	m	Equilibrium level of tank 3
h_4^0	0.6592	m	Equilibrium level of tank 4

Appendix B

Introducción

Este capítulo tiene como fin poner en contexto la tesis desarrollada. Para ello, en primer lugar se presenta la relevancia que, en el campo de la industria, tiene el problema del control de sistemas sometidos a amplios cambios en el punto de funcionamiento. A continuación se hace un breve balance de las estrategias de control planteadas para la solucionarlo, enfocándose en la estrategia que se considera adecuada para abordarlo: el control predictivo. Seguidamente se hace un resumen de los controladores predictivos y se presenta de forma sucinta el problema de la estabilidad con restricciones y cómo el control predictivo soluciona dicha problemática. Para finalizar se presentan los problemas de control que se pretenden resolver con el controlador predictivo propuesto en esta tesis.

B.1 Motivación y objetivos de la tesis

La forma de operar procesos en la industria ha experimentado avances significativos durante los últimos años, guiados por la necesidad de producir de forma segura, limpia y en condiciones competitivas, productos que satisfagan las necesidades del mercado, tanto en cuanto a demanda como en cuanto a calidad y uniformidad. Dos razones justifican este fenómeno: de un lado, la necesidad de dar respuesta a un mercado que en función de sus hábitos sociales y/o culturales se encuentra cada vez más diversificado y exige, además, productos sujetos a estrictos controles de seguridad, variedad y calidad. De otro lado, la necesidad de propiciar un crecimiento sostenible minimizando tanto el impacto medioambiental como el consumo de recursos. Ambos factores contribuyen a que se desee producir de una más eficiente satisfaciendo las exigencias y límites impuestos a los productos.

Por lo tanto resulta deseable buscar técnicas que control que proporcionen leyes que optimicen ciertos criterios de eficiencia garantizando al mismo tiempo la satisfacción de los límites impuestos a los productos. Una de las pocas técnicas que permiten resolver este problema es el control predictivo (Camacho and Bordons, 2004).

En la industria de procesos es habitual la existencia de un punto de operación óptimo o punto de funcionamiento en el cual el proceso debería permanecer con el fin de maximizar su eficiencia. Sin embargo, muchos procesos a lo largo de su normal funcionamiento se ven sometidos a frecuentes cambios en su punto de funcionamiento, de forma que para éstos no existe un punto de funcionamiento, sino más bien un rango de puntos de funcionamiento en cualquiera de los cuales el proceso puede operar durante un período de tiempo. La selección del punto de operación dentro de este rango se hará conforme a la diversidad de productos, lotes o situaciones en las que la planta se pueda encontrar.

La finalidad de este trabajo es el desarrollo de una estrategia de control avanzado de procesos con puntos de operación cambiantes en presencia de restricciones que permitan una operación eficiente, flexible e integral de forma que, haciendo un uso racional de los recursos disponibles, se garantice de manera uniforme la seguridad y calidad del producto.

B.1.1 El control de plantas con puntos de operación cambiantes

El problema de control que se acaba de plantear se caracteriza por dos aspectos que los condicionan. El primer aspecto se deriva del amplio rango de operación que presentan las plantas, el cual acentúa la naturaleza no lineal de sus dinámicas (implícita en las ecuaciones constitutivas asociadas a los balances de materia, energía y cantidad de movimiento) y el grado de incertidumbre (estructural y paramétrica) asociado a sus representaciones en espacio de estados. Además en este tipo de plantas se caracteriza habitualmente por dinámicas complejas, descritas por sistemas acoplados de ecuaciones algebraicas, diferenciales ordinarias y ecuaciones en derivadas parciales.

A la naturaleza compleja del sistema se añade la presencia de restricciones en su operación. Estas restricciones pueden ser límites en las variables que permiten manipular la plantas, así como límites impuestos sobre variables del proceso y pueden derivar de límites físicos de las variables o bien de límites en las zonas de evolución de la planta por motivos económicos, medioambientales o de operación. La presencia de restricciones condiciona de forma notable el comportamiento de los sistemas acentuando su aspecto no lineal, y pueden ser responsables de pérdidas de rendimiento, mal funcionamiento de la planta e incluso inestabilidad (Mayne, 2001).

La forma tradicional de resolver este problema consiste en el diseño de una estructura de control multi-nivel (Tatjewski, 2008). Esta estructura suele ser una estructura jerárquica en el que un control a bajo nivel se encarga del control regulatorio de la planta, generalmente realizada por PIDs o autómatas programables interconectados en red. Por encima de éste se encuentra el control de alto nivel en el que se implementa una estrategia avanzada de control, generalmente multivariable. El controlador de alto nivel determina las consignas de los controladores a bajo nivel para mantener el sistema en el punto de operación deseado. Este punto de operación se determina por un nivel de control superior en el que se implementa un optimizador en tiempo real de las consignas, de acuerdo con los datos de la planta y en base a criterios económicos, provenientes del sistema de integración de información de la planta (CIM). Esta estructura se ilustra en la figura B.1. El control de alto nivel se suele diseñar para

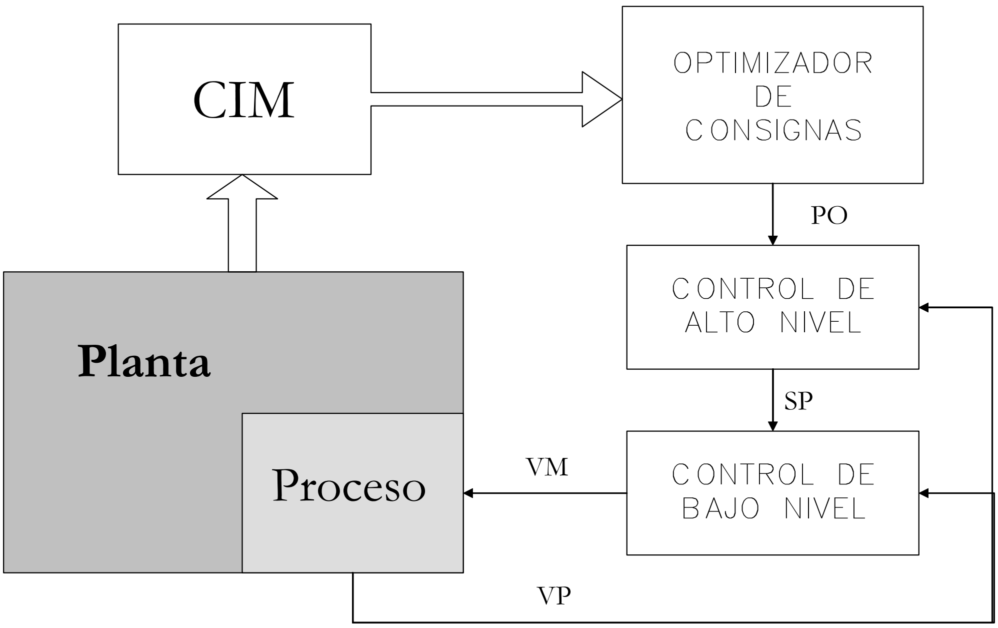


Figure B.1: Estructura de control jerárquico.

regular el sistema en torno al punto de operación y evitar la violación de las restricciones. Cuando el alto nivel indica un cambio de operación el controlador debe hacer frente a esta contingencia, conduciendo el sistema hacia el nuevo punto de operación. Esta operación no es trivial, ya que pueden aparecer problemas ya sea por el cambio de dinámica en el nuevo punto o bien por garantizar la satisfacción de las restricciones en el transitorio al nuevo punto. Con el fin de gestionar cambios significativos en los puntos de operación, el control a alto nivel se suele dividir en dos subniveles (Becerra et al., 1998): un subnivel inferior encargado de regular el sistema y un subnivel superior encargado de la adaptación del controlador al nuevo punto. Este esquema se ilustra en la figura B.2 Dentro de este esquema se enmarcan por ejemplo los controladores adapta-

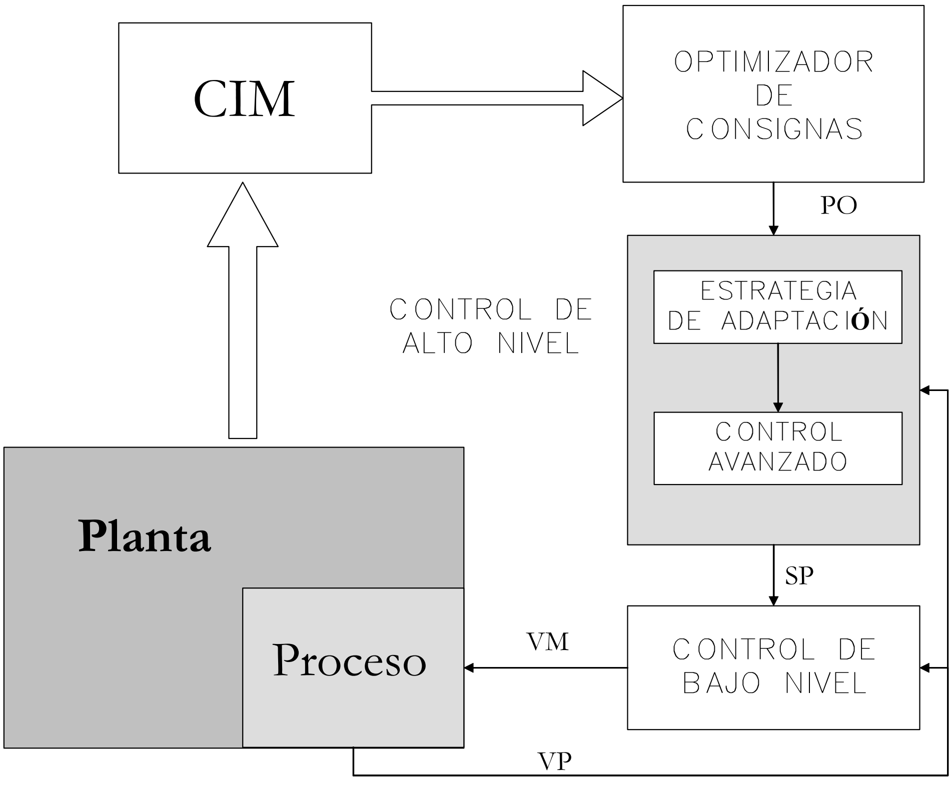


Figure B.2: Estructura de control jerárquico con alto nivel adaptativo.

tivos (como el clásico *gain scheduling* de los aviones al variar la altura de vuelo). Otros controladores de este tipo son los denominados controladores de referencias o *reference governors* (Gilbert et al., 1994, 1999). Estos controladores corresponden al subnivel superior y asumen que en el subnivel inferior se encuentra un controlador avanzado que estabiliza la planta. Los controladores de referencias tienen como fin gestionar de forma racional las referencias de un proceso con el fin de evitar la violación de restricciones cuando el valor deseado de la consigna cambia. Es de alguna forma una sofisticación del conocido filtro de referencias con el fin de evitar la violación de restricciones. El diseño de este tipo de controladores se hace sin atender a la eficiencia del proceso y con el único fin de evitar la violación de los límites. Este aspecto se trata en (Bemporad et al., 1997). Los controladores de referencias también se han extendido con éxito al caso de dinámicas no lineales (Bemporad, 1998b; Angeli and Mosca, 1999; Gilbert and Kolmanovsky, 2002).

Una de las estrategias de control avanzado que más éxito han tenido en la industrial de procesos ha sido el control predictivo (Qin and Badgwell, 1997) pues incorpora un criterio óptimo y restricciones en la ley de control. En el caso del control predictivo existen formulaciones orientadas a gestionar grandes transiciones. Estos controladores permiten grandes cambios en el punto de operación y determinan las acciones de control en base a un criterio de desempeño. Sin embargo, la garantía de estabilidad se basa en una estructura jerárquica como la que se muestra en B.2 en la que el subnivel superior se encarga de conmutar entre el controlador predictivo y el otro controlador orientado a recuperar al sistema en caso de pérdida de factibilidad.

Otra forma de abordar este problema es el control integral en el que se estudian estrategias de control avanzado (generalmente control predictivo) en el que se incorporan objetivos económicos asociados a los cambios de operación. Por lo tanto, parte de la tarea de la optimización del proceso se traslada del optimizador de consignas, con el fin de incorporar de alguna forma el coste asociado a las transiciones en la determinación del punto de operación. En (Becerra and Roberts, 1996; Becerra et al., 1997, 1998) se plantean diferentes alternativas para la integración en el control predictivo con optimización en línea de objetivos económicos, como solución de un problema de control multiobjetivo, donde se minimizan tanto los objetivos de regulación como los económicos. En (Vesely et al., 1998) se plantean los principios y propiedades básicas de un método factible para optimización de estado estacionario de sistemas complejos de los dos niveles de un controlador jerárquico, de modo que el problema se resuelve a través de la resolución de ecuaciones algebraicas. Sin embargo, en la mayoría de estos trabajos no se lleva a cabo un estudio de la estabilidad, robustez y convergencia de los esquemas desarrollados, ni del efecto de la interacción entre componentes.

En consecuencia, las estructuras jerárquicas con garantía de estabilidad y satisfacción de restricciones producen un peor desempeño que un control integral debido a su diseño independiente. Por otro lado las estructuras de control integral adolecen de estudios de estabilidad y satisfacción de restricciones. Por ello resulta deseable diseñar estrategias de control que permitan unificar la solución de este problema de control integral para grandes transiciones en un sólo nivel que garantice la estabilidad en presencia de restricciones y tenga en cuenta la optimización de criterios de desempeño.
El control predictivo es una de las pocas estrategias que permite el control de sistemas con restricciones atendiendo a un criterio óptimo y garantizando la estabilidad y con-

vergencia al punto de equilibrio (Camacho and Bordons, 2004; Mayne, 2001; Rawlings and Mayne, 2009). Por ello, se propone utilizar el control predictivo como estrategia para abordar el problema que se propone. En la figura B.3 se observa que los dos niveles de control de la estructura jerárquica se reemplazan por un sólo controlador predictivo que realiza simultáneamente la tarea de la estabilización y del control al nuevo punto de consigna.

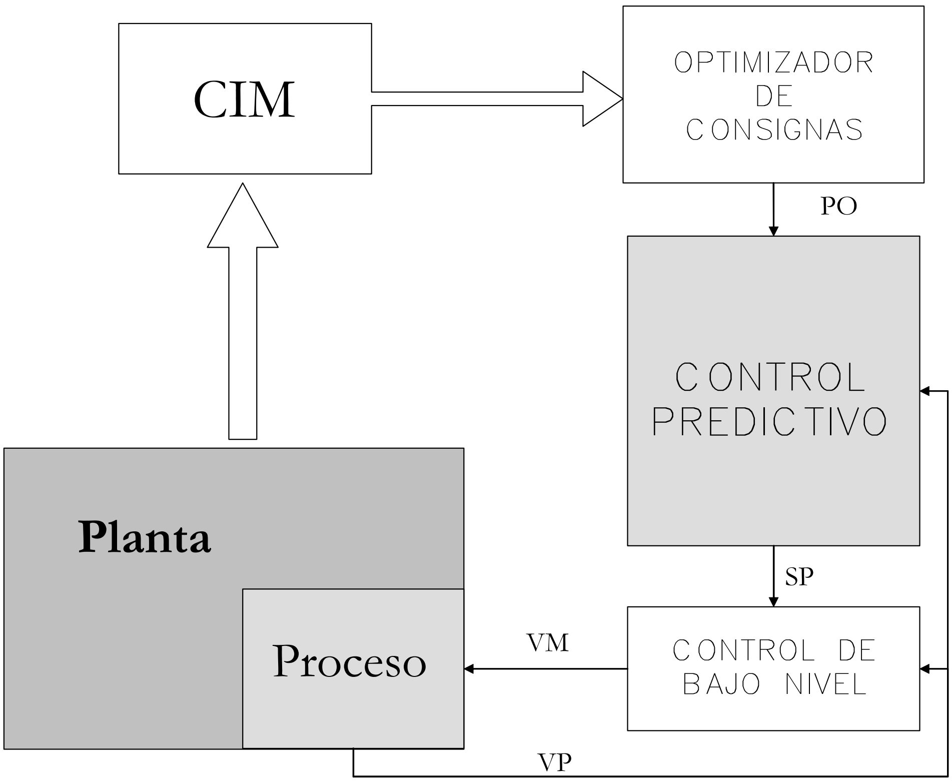


Figure B.3: Estructura de control integral.

El control predictivo basado en modelo ha concentrado el esfuerzo de numerosos investigadores en los últimos años, avanzando notablemente las bases teóricas, la comprensión del problema de control, el estudio de sus características y limitaciones y procedimientos de diseño estabilizante (Mayne et al., 2000; Rawlings and Mayne, 2009; Limon, 2002). Además el control predictivo ha demostrado ser también una técnica efectiva para el control robusto con restricciones (Mayne et al., 2000; Limon, 2002; De Nicolao et al., 1996; Magni et al., 2001c; Fontes and Magni, 2003; Limon et al., 2005, 2006a). Como es bien sabido, el diseño estabilizante de los controladores se basa en el

cálculo de regiones invariantes (Blanchini, 1999; Bertsekas, 1972).
En el caso de sistemas lineales con o sin incertidumbres, existen controladores eficientes que permiten controlar la planta con garantía de estabilidad y satisfacción de restricciones. En este caso se han propuesto técnicas para simplificar el problema de optimización a resolver (Alamo et al., 2005) y permita una implementación eficiente en línea (Bemporad et al., 2002). Por otro lado también se han desarrollado técnicas para el cálculo explícito de la ley de control vía la resolución de un problema multiparamétrico.
En el caso de sistemas no lineales, el problema es más complejo y requiere la solución de un problema de optimización no lineal (Camacho and Bordons, 2004). Para relajar la carga computacional se han establecido condiciones para garantizar la estabilidad en caso de soluciones subóptimas (Scokaert et al., 1999). El control predictivo robustos para sistemas no lineales ha madurado mucho recientemente (Magni et al., 2001c; Limon et al., 2006a, 2009a), pero su complejidad computacional hace que se considere un problema aún sin cerrar. En este sentido la aplicación de técnicas garantistas como (Limon et al., 2005) resultan prometedoras.

B.2 El control predictivo basado en el modelo

La idea básica del control predictivo es la optimización de un coste relacionado con el comportamiento dinámico del sistema (Rawlings and Mayne, 2009), en el que se penaliza tanto el error respecto al punto de equilibrio como el esfuerzo de control necesario para alcanzar dicho equilibrio. Este coste se basa en la predicción de la evolución futura del sistema, que se determina por medio de un modelo de tipo:

$$x(j+1) = f(x(j), u(j))$$

El coste que se pretende optimizar suele ser dado por:

$$V_N(x;\mathbf{u}) = \sum_{j=0}^{N-1} \ell(x(j), u(j)) + V_f(x(N)),$$

donde $u(j)$ es la secuencia de acciones de control futuras calculadas en el instante k, y $x(j)$ es la predicción de los estados futuros del sistema calculada en el instante k, teniendo en cuenta que $x(0) = x$.
La función $\ell(x,u)$ se denomina coste de etapa, mientras que la función $V_f(x(N))$ se denomina coste terminal.

La ley de control se obtiene optimizando el funcional de coste: la secuencia óptima de acciones de control futuras $\mathbf{u}^0$ es justamente la que minimiza el coste $V_N(x;\mathbf{u})$, sin violar las restricciones. El problema de optimización es un problema de programación matemática y se puede escribir de esta forma:

$$
\begin{aligned}
&\min_{\mathbf{u}} V_N(x,\mathbf{u}) \\
&s.t. \\
&\quad x(0) = x, \\
&\quad x(j+1) = f(x(j), u(j)), \\
&\quad u(j) \in U \qquad\qquad\qquad\qquad j = 0, \cdots, N-1 \\
&\quad x(j) \in X \qquad\qquad\qquad\qquad j = 0, \cdots, N-1 \\
&\quad x(N) \in \Omega.
\end{aligned}
$$

La restricción en el estado terminal $x(N)$ se añade a fin de garantizar estabilidad.

En el control predictivo, la realimentación se obtiene utilizando la técnica conocida como *horizonte deslizante*: se aplica al sistema el primer elemento de la secuencia óptima $\mathbf{u}^0$, y la optimización se vuelve a repetir cada tiempo de muestreo. Le ley de control es:

$$h(x) = u^0(0;x)$$

La ley de control obtenida en un controlador predictivo surge de la optimización de un criterio relacionado con el comportamiento del sistema, en el que se penaliza tanto el error respecto al punto de equilibrio como el esfuerzo de control necesario para alcanzar dicho equilibrio. Contrariamente a lo que dicta el sentido común, el hecho de que la actuación aplicada sea óptima no garantiza que el sistema en bucle cerrado alcance el punto de equilibrio tal y como se desea. El problema de la estabilidad tiene su origen en el desarrollo propio de los controladores predictivos: la necesidad de utilizar un horizonte de predicción finito e invariante en el tiempo y la estrategia de horizonte deslizante. Por lo tanto, para evitar este problema, es preciso tener en cuenta algunas condiciones en el diseño del controlador (Mayne et al., 2000; Rawlings and Mayne, 2009).

En la literatura del control predictivo existen diferentes formulaciones estabilizantes:

- MPC con restricción terminal de igualdad: fue propuesto para garantizar estabilidad del problema LQR con restricciones en (Kwon and Pearson, 1977) y extendido en (Keerthi and Gilbert, 1988) a sistemas no lineales. La estabilidad se garantiza imponiendo como restricción terminal

$$x(N) = x_s^*$$

donde x_s^* representa el estado de equilibrio deseado. En el caso de regulación al origen, esta condición es equivalente a:

$$x(N) = 0$$

En (Mayne and Michalska, 1990), se formula este controlador para sistemas en tiempo continuo y se relajan las condiciones para garantizar la estabilidad. En (Chisci et al., 1994; Bemporad et al., 1995) se extiende esta condición a sistemas lineales descritos por un modelo CARIMA, sin restricciones. En este caso, la restricción terminal se traduce en una condición sobre las salidas y las entradas del sistema.

- MPC con coste terminal: la estabilidad se logra incorporando en la función de coste, un término que penalice el estado terminal mediante el denominado coste terminal (Bitmead et al., 1990; Rawlings and Muske, 1993).

- MPC con restricción terminal de desigualdad: la restricción terminal de igualdad se relaja, extendiendo la restricción terminal a una vecindad del origen. Así, se establece una restricción terminal de desigualdad de la forma

$$x(N) \in \Omega$$

siendo el conjunto Ω el denominado conjunto terminal. Esta estrategia fue propuesta en (Michalska and Mayne, 1993) para sistemas no lineales en tiempo continuo y sujeto a restricciones. En este trabajo, se elige como región terminal un invariante positivo del sistema no lineal controlado por un controlador local. Además, para garantizar la factibilidad se introduce como variable de decisión el horizonte de predicción. El controlador así formulado garantiza que conduce al sistema a la región terminal, donde el sistema pasa a regularse por el controlador local que lo estabiliza al origen. De ahí que este controlador se denomine controlador MPC dual. Las bondades de esta formulación son tan notables, que marcó las futuras líneas de investigación en estabilidad.

- MPC con coste y restricción terminal: esta es la estructura en la que se enmarcan las más recientes formulaciones del MPC. El primer trabajo en el que se garantiza estabilidad incorporando ambos ingredientes es en (Sznaier and Damborg, 1987) en el cual, para sistemas lineales sujetos a restricciones politópicas, se considera como controlador local el LQR y como región terminal un invariante asociado. En este trabajo se demuestra que para cada estado, existe un horizonte de predicción suficientemente largo, tal que la solución óptima garantiza la satisfacción de la restricción terminal, lo que permite eliminarla. En (De Nicolao et al., 1998) se propone como coste terminal el coste infinito incurrido por el sistema controlado por el controlador local. En (Magni et al., 2001a), se propone una formulación implementable del controlador predictivo anterior. Se basa en considerar como función de coste terminal una aproximación truncada del coste infinito. Pero lo más destacable de este trabajo es que considera un horizonte de predicción mayor que el de control gracias a la incorporación del controlador local.

Todas las formulaciones de control predictivo con estabilidad garantizada se analizan en (Mayne et al., 2000). En este trabajo se analizan las formulaciones existentes de controladores predictivos con estabilidad garantizada y se establece que el control predictivo con coste terminal y restricción terminal puede, bajo ciertas condiciones, estabilizar asintóticamente un sistema no lineal sujeto a restricciones. Además se establecen condiciones suficientes sobre la función de coste terminal y la región terminal para garantizar dicha estabilidad. Estas condiciones son las siguientes:

- La región terminal Ω debe ser un conjunto invariante positivo admisible del sistema. Es decir, que debe existir una ley de control local $u = h(x)$ tal que estabiliza el sistema en Ω y además la evolución del sistema y las actuaciones en dicho conjunto son admisibles.

- El coste terminal $V(x)$ es una función de Lyapunov asociada al sistema regulado por el controlador local, tal que

$$V(f(x, h(x))) - V(x) \leq -\ell(x, h(x))$$

para todo $x \in \Omega$. Por lo tanto, la ley de control local estabiliza asintóticamente el sistema.

Bajo estas condiciones se garantiza que el coste óptimo es una función de Lyapunov: el hecho de que Ω sea un conjunto invariante garantiza la factibilidad del controlador en todo instante, mientras que la condiciones sobre el coste terminal garantizan la convergencia. Todo esto garantiza la estabilidad asintótica del sistema en bucle cerrado con restricciones.

B.2.1 El problema del cambio de referencia en el contexto de los controladores predictivos

Como se comentó previamente, en la estructura de control industrial un optimizador de consignas es el responsable de proveer el punto de trabajo de la planta, de tal forma que si el punto de trabajo cambia, el controlador a bajo nivel debe realizar dicho cambio. La solución clásica es trasladar el problema al nuevo punto de trabajo (Muske and Rawlings, 1993). En ausencia de restricciones, esta solución siempre es válida. Cuando el problema presenta restricciones, pueden aparecer problemas de perdida de factibilidad o estabilidad. Por lo tanto es necesario un estudio adecuado de este problema.

Un control óptimo con horizonte infinito que considere restricciones podría ser una solución válida, pues realizaría el cambio de referencia de forma admisible. Pero este tipo de controladores no son implementables debido a que un horizonte infinito implica un número infinito de variables de decisión. Así que habrá que utilizar un control predictivo con horizonte finito para resolver el problema. El problema con este tipo de controladores es que un cambio de referencia puede producir una perdida de la factibilidad del problema de optimización por una de las siguientes causas: (i) la restricción terminal calculada para un cierto punto de equilibrio puede no ser un invariante admisible para el nuevo punto de equilibrio, lo que puede provocar la perdida de factibilidad y (ii) la region terminal para el nuevo punto de operación podría no ser alcanzable en N pasos, lo que haría de nuevo, perder la factibilidad del problema. Esto requeriría el recálculo del horizonte para recuperar la factibilidad, por lo que un cambio de referencia conllevaría el rediseño on-line del controlador, lo que no será siempre posible.

Ejemplo B.1 *Considérese el siguiente sistema LTI:*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.0 & 0.5 \\ 1.0 & 0.5 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

sujeto a las siguientes restricciones en el estado y en el control: $\|x\|_\infty \leq 5$ *y* $\|u\|_\infty \leq 0.3$. *Tómese un controlador predictivo con matrices de ponderación del coste de etapa* $Q = I_2$ *y* $R = I_2$.

En la figura B.4 se muestra el problema de la perdida de la factibilidad cuando se produce un cambio de referencia. Supongamos que la planta se encuentra en el punto x_0 *y el estado de referencia es* r_1, $O_\infty(r_1)$ *es el máximo invariante para el sistema controlado por la ley de control* $u = K(x - x_1) + u_1$ *((*x_1, u_1*) es el estado y la acción de control del sistema en equilibrio en el estado de referencia* r_1*). Dicho conjunto es* *también la restricción terminal para nuestro MPC con horizonte $N = 3$, luego la región de atracción será $X_3(r_1)$. Supóngase que en este instante se cambia la referencia al punto r_2. La region terminal ya no es válida puesto que $O_\infty(r_1)$ trasladado al punto de referencia r_2 es un invariante no admisible (las restricciones serían violadas claramente) lo que nos lleva a una posible perdida de factibilidad. Por otro lado el punto x_0 no pertenece a la región de atracción $X_3(r_2)$ con horizonte $N = 3$. Habría que cambiar el horizonte a $N = 6$ para recuperar la factibilidad.*

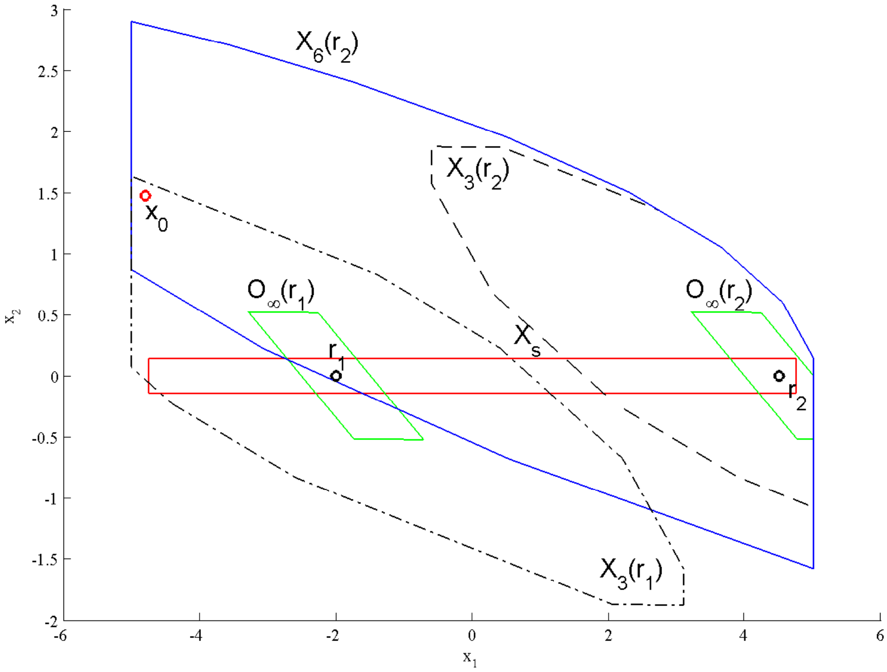


Figure B.4: Pérdida de la factibilidad debido a una región terminal no admisible o a un horizonte insuficiente.

Resumiendo, un cambio de referencia puede producir la perdida de la factibilidad debido a una región terminal no admisible o a un horizonte insuficiente.

B.2.2 El control predictivo y el problema del seguimiento de referencias

Con objeto de superar estos problemas se han propuesto varias soluciones, en (Rossiter et al., 1996; Chisci and Zappa, 2003) se utiliza un controlador auxiliar que es capaz de recuperar la factibilidad en tiempo finito cuando esta de pierde por un cambio de referencia, es pues una estrategia de conmutación. Otra posible solución es la propuesta por (Pannocchia and Kerrigan, 2005; Pannocchia, 2004) considerando el cambio de referencia como una perturbación a rechazar. De esta forma este controlador es capaz de llevar el sistema al punto deseado pero solo cuando las variaciones en la referencia son pequeñas. Es por lo tanto, una solución conservadora.

Un enfoque diferente es el que se le da a este problema en el contexto de los controladores de referencia (Gilbert et al., 1999; Bemporad et al., 1997). Esta técnica de control asume que el sistema está estabilizado de forma robusta por un controlador local, y se diseña un filtro de referencia no lineal para la satisfacción robusta de las restricciones. Este tipo de controlador es capaz de seguir los cambios de referencia sin considerar el desempeño y la región de atracción de dicho controlador. En (Findeisen et al., 2000) se propone un controlador predictivo para familias de puntos de operación basado en pseudolinearización y en una parametrización de los puntos de equilibrio.

En (Limon et al., 2008a) se propone una nueva formulación del control predictivo para seguimiento de referencia, que permite mantener la factibilidad para cualquier cambio de punto de operación. Las principales características de este controlador son: un punto de equilibrio artificial considerado como variable de decisión, un coste que penalice la distancia entre la trayectoria predicha y el punto de equilibrio artificial, un coste adicional que penalice la distancia entre el punto de equilibrio artificial y el punto de equilibrio deseado, llamado *coste de offset*, y una restricción terminal extendida, el conjunto invariante para tracking. Este controlador garantiza estabilidad y factibilidad recursiva para cualquier cambio de referencia. Este controlador soluciona el problema de fectibilidad y estabilidad del MPC ante cambios de punto de operación, pero el desempeño del sistema controlado puede verse afectado (Alvarado, 2007).

En el caso de sistemas no lineales, el problema de cambio de referencia ha sido tratado en (Magni et al., 2001a, 2002; Magni and Scattolini, 2005). En particular, en (Magni et al., 2001a) se presenta un algoritmo para sistemas no lineales basado en realimentación de la salida, capaz de resolver el problema del seguimiento de señales exógenas, rechazando perturbaciones asintóticamente. En (Magni and Scattolini, 2005) se presenta un algoritmo caracterizado por la presencia de un integrador antes del sistema, a fin de garantizar la solución del problema de seguimiento asintóticamente.

B.3 Formulaciones robustas de control predictivo

La forma habitual de considerar las incertidumbres en el control predictivo es incorporando todas las posibles realizaciones de éstas en la solución del problema de optimización. Nótese que las restricciones en la evolución de los estados se deben satisfacer de una forma robusta, es decir, para todas las posibles realizaciones de las incertidumbres. La incorporación de restricciones en el estado complica notablemente el problema, pero, aún en el caso en el que no haya restricciones sobre los estados, la restricción terminal siempre está presente en el problema de optimización pues se añade para garantizar la estabilidad del controlador.

El coste a optimizar puede basarse en las predicciones nominales del sistema o bien considerar el efecto de las incertidumbres tomando, por ejemplo, la peor situación posible. Esto da lugar a la denominada formulación min-max (Fontes and Magni, 2003; Limon et al., 2006a; Mayne, 2001). Otra formulación consiste en añadir un término en la función de coste de etapa que pondera la posible incertidumbre, como en la formulación H_∞.

Esta forma de considerar las incertidumbres es intuitiva y razonable, pero puede conducir a soluciones muy conservadoras. Este conservadurismo radica en la naturaleza misma del control predictivo: la predicción en bucle abierto. Este aspecto da lugar a la denominada *formulación en bucle cerrado* y fue introducida en (Scokaert and Mayne, 1998; Lee and Yu, 1997) en el contexto del min-max. En esta formulación, el problema de control no está planteado en términos de una secuencia de actuaciones, sino de una secuencia de leyes de control lo cual hace que el problema de optimización implicado sea infinito-dimensional. En consecuencia, estos controladores constituyen una herramienta meramente teórica (Mayne et al., 2000).

Dentro del control predictivo en bucle cerrado se pueden considerar otras formulaciones como por ejemplo el trabajo presentado en (Kothare et al., 1996). En éste se propone un controlador que estabiliza una planta incierta tal que se puede expresar en cada instante como una combinación convexa de una serie de plantas lineales y que presenta restricciones en los estados y en las actuaciones. En esta formulación se considera como variable de optimización un controlador lineal que estabiliza todas las plantas y se puede plantear como un LMI que se resuelve en cada instante.

También se pueden considerar dentro de los controladores en bucle cerrado los trabajos (Bemporad, 1998a) y (Chisci et al., 2001) en los cuales se parametriza la ley de control con una ley de control que estabiliza la planta nominal. El controlador predictivo por lo tanto se formula con predicciones nominales y restricciones politopicas que se contraen a lo largo del horizonte de predicción. Recientemente en (Mayne et al., 2005) se ha aprovechado esta idea para diseñar un controlador robusto prealimentado basado en la noción de tubo (Langson et al., 2004). Además, en (Chisci and Zappa, 1999) se añade una restricción adicional con el fin de garantizar la satisfacción robusta de las incertidumbres. Esta idea se generaliza al caso no lineal en (Kerrigan, 2000), donde se analiza utilizando la teoría de conjuntos invariantes la satisfacción robusta de las restricciones.

En (Limon et al., 2009a) se presenta el concepto de estabilidad entrada estados como técnica para la análisis de estabilidad robusta de controladores predictivos, y se estudian diferentes controladores existentes en literatura.

B.4 El control predictivo por zonas

En muchos casos, el punto de operación económico no está dado por un punto fijo, sino más bien por una region en la cual la salida del sistema debería permanecer la mayoría del tiempo. En general, debido a las necesidades en las operaciones, las salidas de un proceso se pueden distinguir en dos categorías: (i) controladas en un punto de operación, (ii) salidas controladas en un intervalo de operación. Por ejemplo, el ritmo de producción o la calidad de producto pueden caer en el primer grupo, mientras que variables de proceso, como temperaturas, presiones, o niveles, se configuran en la segunda categoría. Las razones que determinan la importancia de trabajar con intervalos de operación, están relacionadas con los grados de libertad del proceso: en general, las zonas de operación aparecen siempre cuando no hay suficientes variables manipulada respeto a las controladas. A nivel de concepto, los intervalos en las salidas de procesos no se deben de interpretar como restricciones en las salidas, pues se trata de condiciones de equilibrio que se pueden despreciar temporalmente, mientras que las restricciones se deben siempre satisfacer. Además, la determinación de zonas de salidas está relacionada con la determinación del punto de operación de equilibrio del proceso, y por lo tanto no se trata de un problema sencillo. La compatibilidad del conjunto de entradas admisibles y de salidas deseadas necesita también un cuidado particular. En (Vinson and Georgakis, 2000) y (Lima and Georgakis, 2008), por ejemplo, se define un índice de operabilidad, que determina qué parte de la región de salida se puede alcanzar con el conjunto de entradas disponible, teniendo en cuenta las posibles perturbaciones que se presenten. En la practica, los operadores de plantas de proceso, no están al tanto de la presencia de estas zonas de operación, y puede ocurrir que se escojan zonas no admisibles o no alcanzables.

En la literatura, se ha propuestos diferentes enfoques para el problema del control por zonas. (Qin and Badgwell, 2003) describe diferentes controladores industriales que siempre permiten la opción del control por zonas. En este trabajo, se presentan dos maneras de solucionar el problema del control por zonas: 1) definiendo restricciones blandas, y 2) usando la aproximación de punto de equilibrio de restricciones blandas para determinar las cotas superiores e inferiores de los intervalos (el conocido algoritmo DMC-plus). El principal problema de esos controladores industriales es la falta de pruebas de estabilidad. Un segundo ejemplo de control por zonas es el trabajo presentado en (Zanin et al., 2002), cuyo enfoque de control se aplica a un sistema FCC. La estrategia de control propuesta en este trabajo presenta buenas prestaciones, pero para ella no se puede demostrar estabilidad, pues el sistema de control conmuta continuamente de un controlador a otro. Un controlador predictivo por zonas, estable en bucle cerrado se presenta en (Gonzalez and Odloak, 2009). En este trabajo se presenta un controlador incorpora puntos de equilibrio económicos de referencia en el funcional de coste, teniendo en cuenta la presencia de zonas de salida. La clásica demostración de estabilidad se extiende al caso de las zonas de operación, asumiendo estabilidad del sistema en bucle abierto. Al problema de optimización se añade un conjunto de variables de holgura, para garantizar convergencia y factibilidad recursiva. Este controlador está diseñado para sistemas estables en bucle abierto, y no permite obtener optimalidad local, puesto que se considera un controlador local nulo. Una extensión de esta estrategia de control al caso robusto se presenta en (González et al., 2009).

B.5 Optimización de prestaciones económicas

Como se comentó previamente, la estructura estándar de los sistemas de control industrial se caracteriza por la presencia de dos niveles. El primer nivel realiza una optimización para el calculo del punto de operación y se suele denominar de Optimizador en Tiempo Real (RTO). El RTO determina el punto de operación óptimo y lo envía a la segunda capa, el sistema de control avanzado, donde se realiza la optimización dinámica. En muchos procesos, el control predictivo es la estrategia de control que se utiliza para este nivel (Rawligns and Amrit, 2009).

Los problemas de esta estructura jerárquica se relacionan con el papel que tiene el RTO. Las optimizaciones que se realizan en esta capa se suelen basar en un modelo estático de la planta. En cada instante de muestreo se optimiza un criterio económico, con el objetivo de hallar el mejor punto de equilibrio para el modelo estático. El resultado de la optimización se envía al controlador avanzado como punto de operación. El problema es que este punto de operación suele ser inconsistente o no alcanzable, y eso ocurre por las inconsistencias entre el modelo estacionario del RTO y el modelo dinámico del usado para la regulación. En (Rao and Rawlings, 1999) los autores proponen métodos para resolver est problema y hallar el punto de equilibrio mas cercano al punto de operación no alcanzable determinado por el RTO.

Un tema muy estudiado últimamente en la comunidad académica es el diseño de controladores económicos. Estos controlador se definen de esta manera en cuanto la optimizaciones que realiza el RTO no están basadas en el modelo dinámico del sistema, si no más bien en un criterio económico que tenga en cuenta de la demanda de producción del sistema. Por lo tanto, los punto de operación óptimos que el RTO calcula no tienen porque coincidir con el punto de equilibrio dinámico del sistema (Kadam and Marquardt, 2007).

En (Rawligns and Amrit, 2009) además se subrayan las ventajas de realizar las optimizaciones económicas en el mismo controlador de alto nivel, en particular un controlador predictivo. Los autores primero estudian el caso de puntos de operación no alcanzables y demuestran cómo muchas veces lo óptimo no es alcanzar rápidamente el punto de equilibrio. Además, en este trabajo se considera el caso de substituir el clásico funcional de coste del predictivo, con un funcional que minimice algún criterio económico. En (Rawlings et al., 2008) se presenta un controlador predictivo estable para el caso de puntos de operación no alcanzable. (Würth et al., 2009) propone un controlador económico a horizonte infinito que garantiza estabilidad. En (Diehl et al., 2011) y (Huang et al., 2011) se proponen unos controladores económicos que demuestran estabilidad usando funciones de Lyapunov.

B.6 Sistemas de gran escala

Las plantas en la industria de procesos suelen ser sistemas de gran escala, caracterizados por diferentes unidades interconectadas entre ellas. Por lo tanto, esas plantas se pueden dividir en diferentes subsistemas que comunican entre ellos por medio de redes de distinta naturaleza, sea por ejemplo de materiales, energías y flujos de informaciones (Stewart et al., 2010). El control total de esas plantas usando controladores centralizados - un solo agente controlando todos los subsistemas - es difícil de realizarse. No se trata sólo de un problema computacional. De hecho hoy en día, la gran potencia computacional de las maquinas, así como la existencia de algoritmos de optimización rápidos, hacen que el control centralizado sea una tarea realizable incluso para problemas grandes. (Bartlett et al., 2002; Pannocchia et al., 2007). Puesto que cada subsistema realiza una tarea diferente, a veces en la ejecución de sus operaciones cada subsistema debe despreciar las informaciones de los otros subsistemas. En otras ocasiones, el intercambio de informaciones entre los distintos subsistemas juega un papel importante en las prestaciones óptimas de la planta. Por lo tanto, el verdadero problema es la organización y el mantenimiento del controlador centralizado.

Otra estrategia de control muy utilizada, es el control descentralizado. En este caso, cada subsistema se controla independientemente, sin algún intercambio de información entre los diferentes subsistemas. La información que fluye en la red se considera como una perturbación (Huang et al., 2003; Sandell Jr. et al., 1978). El inconveniente de esta formulación son las perdidas de información cuando la conexión entre subsistemas es muy grande (Cui and Jacobsen, 2002).

Hoy en día, unos de los temas mas discutidos en la comunidad científica del control automático es el control distribuido. Se trata de una estrategia de control basada en diferentes agentes controlando los diferentes subsistemas, que pueden o no intercambiar informaciones entre ellos. En literatura, hay diferentes estrategias de control propuestas. La diferencia entre ellas está en la forma en que se tratan las informaciones. En el control distribuido no cooperativo, cada agente toma decisiones sobre su propio subsistema considerando solo localmente las informaciones de los otros subsistemas (Camponogara et al., 2002b; Dunbar, 2007). Esta estrategia de control se suele denominar también de juego dinámico no cooperativo, y las prestaciones de la planta suelen converger a un equilibrio de Nash (Başar and Olsder, 1999). Los controladores distribuidos cooperativo, por otro lado, consideran el efecto de todas las acciones de control sobre todos los subsistemas de toda la red. Cada agente optimiza un coste global, como por ejemplo un coste centralizado. Por lo tanto, las prestaciones de estos controladores convergen a un equilibrio de Pareto, como en el caso centralizado.

El control predictivo es una estrategia de control muy utilizada también en el marco del control distribuido (Rawlings and Mayne, 2009, Chapter 6). En (Magni and Scattolini, 2006) se presenta un controlador predictivo no lineal, que se caracteriza por la ausencia de intercambio de informaciones entre agentes. Una demostración de estabilidad entrada-estado se provee en (Raimondo et al., 2007b). En (Liu et al., 2009, 2008) los autores presentan un controlador para sistemas en red, basado en control predictivo. En (Venkat et al., 2007; Stewart et al., 2010) se propone una estrategia de control predictivo distribuido cooperativo, caracterizada por un algoritmo de resolución del problema de optimización subóptimo.

B.7 Contribuciones de la tesis

En este trabajo de tesis se ha abordado el análisis y diseño de controladores predictivos con punto de operación cambiante basados en (Limon et al., 2008a). Esta formulación se ha extendido a controladores predictivos óptimos, control por zonas, control distribuido, control predictivo no lineal y económico, como se detalla a continuación.

B.7.1 Control predictivo para tracking con prestaciones óptimas en bucle cerrado

En el capítulo 2 se presenta una mejorada formulación del MPC para tracking (Limon et al., 2008a). El controlador propuesto hereda las características principales de MPC para tracking (Limon et al., 2008a), que son:

- Puntos de equilibrio artificiales considerados como variable de decisión.
- Un funcional de coste que minimiza el error entre estado actual y punto de equilibrio artificial.
- Un término adicional añadido a la función de coste, que penaliza la desviación entre la referencia y la referencia artificial (el coste de offset).
- Se considera como restricción terminal en el estado y en la referencia artificial, un invariante para tracking.

En ese capítulo, el MPC para tracking se extiende considerando un coste de offset genérico. Bajo algunas condiciones suficientes, este coste garantiza la propiedad de la otpimalidad local, proveyendo al controlador prestaciones óptimas en bucle cerrado.

Además, el capítulo presenta una caracterización de la región de optimalidad local y una manera para calcularla de bajo coste computacional.

Esta nueva formulación permite considerar cualquier conjunto de variables de proceso como referencia objetivo, de manera que el controlador resulta apto para plantas no cuadradas. Además, el controlador propuesto se aplica a los casos de puntos objetivos inconsistentes con el modelo de predicción o con las restricciones. En ese caso, el controlador lleva el sistema al punto de equilibrio admisible que minimice el coste de offset.

B.7.2 Control predictivo para control por zonas

En el capítulo 3 se presenta la extensión del MPC para tracking al problema del control por zonas. El controlador se formula como un control en el que se pretende alcanzar un conjunto objetivo, que define las zonas de operación. En particular, se formula un coste de offset que explota el concepto de distancia a un conjunto. El controlador propuesto garantiza factibilidad recursiva y convergencia al conjunto objetivo para cualquier planta estabilizable. Esta propiedad se cumpla para cualquier clase de conjunto objetivo convexo, o variante en el tiempo. En el capítulo además se proponen 3 formulaciones para lidiar con conjuntos objetivos poliédricos, que permiten resolver el problema de optimización en la forma de la programación cuadrática. Una de esas formulaciones permite considerar puntos objetivo o conjuntos objetivo al mismo tiempo.

B.7.3 Control predictivo robusto basado en predicciones nominales

El tema del capítulo 4 es el problema del control predictivo para tracking robusto. Se propone un controlador basado en predicciones nominales. Este controlador es una extensión del controlador presentado en (Ferramosca et al., 2010a) y en el capítulo 3 al caso de la presencia de incertidumbres aditivas. El controlador propuesto explota los resultados presentados en (Chisci et al., 2001). La planta se asume lineal y las incertidumbres aditivas acotadas. El controlador propuesto garantiza factibilidad para cualquier cambio de conjunto objetivo y convergencia a un entorno del objetivo, si este es admisible. En el caso contrario, el controlador lleva el sistema al punto de equilibrio óptimo mas cercano (con respeto al coste de offset).

B.7.4 Control predictivo distribuido para tracking

En el capítulo 5 se propone una estrategia de control predictivo distribuido para sistemas lineales. En particular, el controlador predictivo para tracking presentado en el capítulo 2 se extiende al caso de sistemas distribuidos de larga escala.

Entre las diferentes soluciones presentes en literatura, el capítulo se centra en particular en la formulación del control predictivo distribuido cooperativo presentada en (Rawlings and Mayne, 2009, Capítulo 6), en (Venkat, 2006) y en (Stewart et al., 2010). En esa formulación, los agentes comparten un objetivo de control común, que se puede considerar como el objetivo de control de la planta. Por lo tanto, cada agente calcula su secuencia de acciones de control óptimas minimizando el único funcional de coste de forma distribuida. La estabilidad de los controladores se demuestra por medio de la teoría del control predictivo subóptimo (Scokaert et al., 1999). Convergencia al punto objetivo y factibilidad recursiva se garantizan por medio del calculo centralizado de la referencia y de un especifico algoritmo de inicialización.

B.7.5 Control predictivo para tracking de sistemas no lineales sujetos a restricciones

En el capítulo 6 se trata el problema del diseño de un controlador predictivo para seguimiento en caso de sistemas no lineales sujetos a restricciones.

El controlador propuesto en ese capítulo hereda las características principales del controlador presentado en el capítulo 2. En particular, de particular interés resulta ser el problema del cálculo de los ingredientes terminales. Se proponen tres formulaciones del mismo controlador, respectivamente relativas a los tres casos de restricción terminal de igualdad, restricción terminal de desigualdad, control predictivo sin restricción terminal.

En particular, para la formulación con restricción de igualdad, se propone un método basado en el modelado LDI - inclusiones de diferencias lineales - de las plantas propuesto en (Wan and Kothare, 2003a,b).

B.7.6 Control predictivo económico para objetivos económicos cambiantes

En (Rawlings et al., 2008) y (Diehl et al., 2011) se propone una nueva formulación de control predictivo, que considera un funcional de coste basado en objetivos económicos, en lugar del clásico funcional basado en errores de tracking. En (Rawlings et al., 2008) y (Diehl et al., 2011) los autores demuestran que ese controlador estabiliza el sistema en un punto de equilibrio óptimo desde el punto de vista del criterio económico considerado, y provee mejores prestaciones con respeto al objetivo que los estándar controladores para tracking.

Si el objetivo económico cambia por variación de la demanda, capacidad de producción, etc., el punto de equilibrio óptimo puede que cambie, y en virtud de ello se puede perder la factibilidad del controlador. En el capítulo 7 se presenta un controlador predictivo económico para objetivos econónmicos cambiantes. Ese controlador es una formulación híbrida entre el control predictivo para tracking (Limon et al., 2008a; Ferramosca et al., 2009a) y el controlador predictivo económico (Rawlings et al., 2008; Diehl et al., 2011), dado que hereda la factibilidad garantizada para cualquier cambio del objetivo del primero, y la optimalidad con respeto al objetivo del segundo.

B.8 Lista de publicaciones

B.8.1 Capítulos de libro:

1. D. Limon, A. Ferramosca, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking of constrained nonlinear systems. *Book of the 3^{rd} International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, pp. 315-323. 2009.

2. D. Limon, T. Álamo, D.M. Raimondo, D. Muñoz de la Peña, J.M. Bravo, A. Ferramosca, and E. F. Camacho. Input-to-State stability: a unifying framework for robust Model Predictive Control. *Book of the 3^{rd} International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, pp. 1-26. 2009.

B.8.2 Publicaciones en revista:

1. A. Ferramosca, D. Limon, A.H. González, D. Odloak, and E. F. Camacho. MPC for tracking zone regions. *Journal of Process Control*, 20 (4), pp. 506-516. 2010.

2. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking with optimal closed-loop performance. *Automatica*, 45 (8), pp. 1975-1978. 2009.

3. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, F. Castaño, and E. F. Camacho. Optimal MPC for tracking of constrained linear systems. *International Journal of Systems Science*, 42 (8). Publicación en Agosto 2011.

B.8.3 Publicaciones en congresos internacionales:

1. A. Ferramosca, D.Limon, J.B. Rawlings, and E.F. Camacho. Cooperative distributed MPC for tracking. *Proceedings of the* 18^{th} *IFAC World Congress*, 2011.

2. A. Ferramosca, J.B. Rawlings, D.Limon, and E.F. Camacho. Economic MPC for a changing economic criterion. *Proceedings of* 49^{th} *IEEE Conference on Decision and Control, (CDC)*, 2010.

3. D. Limon, I. Alvarado, A. Ferramosca, T. Álamo, and E. F. Camacho. Enhanced robust NMPC based on nominal predictions. *Proceedings of* 8^{th} *IFAC Symposium on Nonlinear Control Systems, (NOLCOS)*. 2010.

4. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking of constrained nonlinear systems. *Proceedings of* 48^{th} *IEEE Conference on Decision and Control, (CDC)*, 2009.

5. A. Ferramosca, D. Limon, A.H. González, D. Odloak, and E. F. Camacho. MPC for tracking target sets. *Proceedings of* 48^{th} *IEEE Conference on Decision and Control, (CDC)*, 2009.

6. A. Ferramosca, D. Limon, F. Fele, and E. F. Camacho. L-Band SBQP-based MPC for supermarket refrigeration systems. *Proceedings of* 10^{th} *European Control Conference, (ECC)*, 2009.

7. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking with optimal closed-loop performance. *Proceedings of* 47^{th} *IEEE Conference on Decision and Control, (CDC)*, 2008.

8. D. Limon, A. Ferramosca, I. Alvarado, T. Álamo, and E. F. Camacho. MPC for tracking of constrained nonlinear systems. *Proceedings of the* 3^{rd} *International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, 2008.

9. I. Alvarado, D. Limon, A. Ferramosca, T. Álamo, and E. F. Camacho. Robust tube-based MPC for tracking applied to the quadruple tank process. *Proceedings of the IEEE International Conference on Control Applications, (CCA).* 2008.

10. A. Ferramosca, D. Limon, I. Alvarado, T. Álamo, and E. F. Camacho. Optimal MPC for tracking of constrained linear systems. *Proceedings of* 8^{th} *Portuguese Conference on Automatic Control, (CONTROLO).* 2008.

B.8.4 Publicaciones en congresos nacionales:

1. A. Ferramosca, I. Alvarado, D. Limon, and E. F. Camacho. MPC para el seguimiento del ángulo de cabeceo de un helicóptero. *Proceedings of* 28^{th} *Jornadas de Automática.* 2007.

Bibliography

T. Alamo, D. M. de la Peña, D. Limon, and E. F. Camacho. Constrained min-max predictive control: Modifications of the functional leading to polynomial complexity. *IEEE Transactions on Automatic Control*, 50:710–714, 2005.

I. Alvarado. *Model Predictive Control for Tracking Constrained Linear Systems*. PhD thesis, Univ. de Sevilla., 2007.

I. Alvarado, D. Limon, T. Alamo, and E. Camacho. Output feedback robust tube based MPC for tracking of piece-wise constant references. In *Proceedings of 46th IEEE Conference on Decision and Control, CDC 2007*, New Orleans, LA, USA, December, 12-14 2007a.

I. Alvarado, D. Limon, T. Alamo, M. Fiacchini, and E. Camacho. Robust tube-based MPC for tracking of piece-wise constant references. In *Proceedings of 46th IEEE Conference on Decision and Control, CDC 2007*, New Orleans, LA, USA, December, 12-14 2007b.

I. Alvarado, D. Limon, A. Ferramosca, T. Alamo, and E. F. Camacho. Robust tube-based MPC for tracking applied to the quadruple tank process. In *Proceedings of the IEEE International Conference on Control Applications, CCA 2008*, San Antonio, Texas, September, 3-5 2008.

D. Angeli and E. Mosca. Command governors for constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 44:816–820, 1999.

R. A. Bartlett, L. T. Biegler, J. Backstrom, and V. Glopa. Quadratic programming algorithms for large-scale model predictive control. *JPC*, 12:775–795, 2002.

T. Başar and G. J. Olsder. *Dynamic Noncooperative Game Theory*. SIAM, Philadelphia, 1999.

V. M. Becerra and P. D. Roberts. Dynamic integrated system optimisation and parameter estimation for discrete time optimal control of nonlinear systems. *Int. Journal of Control*, 63:257–281, 1996.

V. M. Becerra, P. D. Roberts, and G. W. Griffiths. Process optimisation using predictive control: application to a simulated reaction system. In *Proc. IFAC Symp. On Advanced Control of Chemical Processes*, 1997.

V. M. Becerra, P. D. Roberts, and G. W. Griffiths. Novel developments in process optimisation using predictive control. *Journal of Process Control*, 8:117–138, 1998.

A. Bemporad. Reducing conservativeness in predictive control of constrained systems with disturbances. In *Proceedings of the CDC*, 1998a.

A. Bemporad. Reference governor for constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 43:415–419, 1998b.

A. Bemporad, L. Chisci, and E. Mosca. On the stabilizing property of SIORHC. *Automatica*, 31:2013–2015, 1995.

A. Bemporad, A. Casavola, and E. Mosca. Nonlinear control of constrained linear systems via predictive reference management. *IEEE Transactions on Automatic Control*, 42:340–349, 1997.

A. Bemporad, M. Morari, V. Dua, and E. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38:3–20, 2002.

D. M. Bertsekas. Infinite-time reachability of state-space regions by using feedback control. *IEEE Transactions on Automatic Control*, 17:604–612, 1972.

R. R. Bitmead, M. Gervers, and V. Wertz. *Adaptive optimal control - The thinking's man GPC.* Prentice-Hall, 1990.

F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999.

S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, 2006.

E. F. Camacho and C. Bordons. *Model Predictive Control.* Springer-Verlag, 2nd edition, 2004.

E. Camponogara, D. Jia, B. H. Krogh, and S. Talukdar. Distributed Model Predictive Control. *IEEE Control Systems Magazine*, 22(1):44–52, 2002a.

E. Camponogara, D. Jia, B. H. Krogh, and S. Talukdar. Distributed model predictive control. *IEEE Control Systems Magazine*, pages 44–52, 2002b.

L. Chisci and G. Zappa. Robustifying a predictive controller against persistent disturbances. In *Proceedings of the ECC*, 1999.

L. Chisci and G. Zappa. Dual mode predictive tracking of piecewise constant references for constrained linear systems. *Int. J. Control*, 76:61–72, 2003.

L. Chisci, , and E. Mosca. Stabilizing i-o receding horizon control of CARMA plants. *IEEE Transactions on Automatic Control*, 39:614–618, 1994.

L. Chisci, J. A. Rossiter, and G. Zappa. Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37:1019–1028, 2001.

L. Chisci, P. Falugi, and G. Zappa. Predictive tracking control of constrained nonlinear systems. *IEE Proceedings - Control Theory and Applications*, 152:3:309–316, 2005.

H. Cui and E. W. Jacobsen. Performance limitations in decentralized control. *JPC*, 12:485–494, 2002.

G. De Nicolao, L. Magni, and R. Scattolini. On the robustness of receding horizon control with terminal constraints. *IEEE Transactions on Automatic Control*, 41: 451–453, 1996.

G. De Nicolao, L. Magni, and R. Scattolini. Stabilizing receding-horizon control of non-linear time-varying systems. *IEEE Transactions on Automatic Control*, 43:1030–1036, 1998.

M. Diehl, R. Amrit, and J. B. Rawlings. A lyapunov function for economic optimizing model predictive control. *IEEE Transactions on Automatic Control*, 56(3):703 –707, 2011.

W. B. Dunbar. Distributed Receding Horizon Control of Dynamically Coupled Nonlinear Systems. *TAC*, 52(7):1249–1263, 2007.

A. Ferramosca, D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho. MPC for tracking with optimal closed-loop performance. In *Proceedings of 47th IEEE Conference on Decision and Control, CDC 2008*, Cancun, Mexico, December, 9-11 2008a.

A. Ferramosca, D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho. Optimal MPC for tracking constrained linear systems. In *Proceedings of 8th Portuguese Conference on Automatic Control, CONTROLO 2008*, Vila Real, Portugal, July, 21-23 2008b.

A. Ferramosca, D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho. MPC for tracking with optimal closed-loop performance. *Automatica*, 45:1975–1978, 2009a.

A. Ferramosca, D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho. MPC for tracking constrained nonlinear systems. In *Proceedings of 48th IEEE Conference on Decision and Control, CDC 2009*, Shanghai, China, December, 16-18 2009b.

A. Ferramosca, D. Limon, A. H. González, D. Odloak, and E. F. Camacho. MPC for tracking target sets. In *Proceedings of 48th IEEE Conference on Decision and Control, CDC 2009*, Shanghai, China, December, 16-18 2009c.

A. Ferramosca, D. Limon, A. H. González, D. Odloak, and E. F. Camacho. MPC for tracking zone regions. *Journal of Process Control*, 20:506–516, 2010a.

A. Ferramosca, J. B. Rawlings, D. Limon, and E. F. Camacho. Economic MPC for a changing economic criterion. In *Proceedings of 49th IEEE Conference on Decision and Control, CDC 2010*, 2010b.

A. Ferramosca, D. Limon, I. Alvarado, T. Alamo, F. Castaño, and E. F. Camacho. Optimal MPC for tracking of constrained linear systems. *Int. J. of Systems Science*, 42(8), August 2011a. Accepted for pubblication.

A. Ferramosca, D. Limon, J. B. Rawlings, and E. F. Camacho. Cooperative distributed MPC for tracking. In *Proceedings of the 18th IFAC World Congress*, 2011b.

R. Findeisen, H. Chen, and F. Allgöwer. Nonlinear predictive control for setpoint families. In *Proc. Of the American Control Conference*, 2000.

F. A. C. C. Fontes and L. Magni. Min-max model predictive control of nonlinear systems using discontinuous feedbacks. *IEEE Transactions on Automatic Control*, 48(10):1750–1755, 2003.

E. Gilbert and I. Kolmanovsky. Nonlinear tracking control in the presence of state and control constraints: A generalized reference governor. *Automatica*, 38:2063–2073, 2002.

E. Gilbert, I. Kolmanovsky, and K. T. Tan. Discrete time reference governors and the nonlinear control of systems with state and control constraints. *International Journal of Robust and Nonlinear Control*, 5:487–504, 1999.

E. G. Gilbert and K. Tan. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36:1008–1020, 1991.

E. G. Gilbert, I. Kolmanovsky, and K. T. Tan. Nonlinear control of discrete-time linear systems with state and control constraints: A reference governor with global convergence properties. In *Proceedings of the CDC*, 1994.

A. H. Gonzalez and D. Odloak. A stable MPC with zone control. *Journal of Process Control*, 19:110–122, 2009.

A. H. González, J. L. Marchetti, and D. Odloak. Robust model predictive control with zone control. *IET Control Theory and Application*, pages doi:10.249/iet–cta:20070211, 2009.

G. Grimm, M. J. Messina, S. Z. Tuna, and A. R. Teel. Examples when nonlinear model predictive control is nonrobust. *Automatica*, 40:1729–1738, 2004.

B. Hu and A. Linnemann. Towards infinite-horizon optimality in nonlinear model predictive control. *IEEE Transactions on Automatic Control*, 47(4):679–682, 2002.

R. Huang, E. Harinath, and L. T. Biegler. Lyapunov stability of economically oriented NMPC for cyclic processes. *Journal of Process Control*, 2011. Article in press.

S. Huang, K. K. Tan, and T. H. Lee. Decentralized control design for large-scale systems with strong interconnections using neural networks. *TAC*, 48:805–810, 2003.

A. Isidori. *Nonlinear Control Systems*. Springer, 3 edition, 1995.

K. H. Johansson. The quadruple-tank process. *IEEE Trans. Cont. Sys. Techn.*, 8: 456–465, 2000.

C. N. Jones and J. Maciejowski. Primal-dual enumeration for multiparametric linear programming. *Mathematical Software-ICMS 2006*, pages 248–259, 2006.

C. N. Jones and M. Morari. Polytopic approximation of explicit model predictive controllers. *Automatic Control, IEEE Transactions on*, 55(11):2542–2553, 2010.

C. N. Jones, E. C. Kerrigan, and J. Maciejowski. Lexicographic perturbation for multiparametric linear programming with applications to control. *Automatica*, 43(10): 1808–1816, 2007.

J. V. Kadam and W. Marquardt. Integration of economical optimization and control for intentionally transient process operation. In R. Findeisen, F. Allgöwer, and L. T. Biegler, editors, *International Workshop on Assessment and Future Direction of Nonlinear Model Predictive Control*, pages 419–434. Springer, 2007.

D. E. Kassmann, T. A. Badgwell, and R. B. Hawkings. Robust target calculation for model predictive control. *AIChE Journal*, 45:1007–1023, 2000.

S. S. Keerthi and E. G. Gilbert. Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations. *Journal of Optimization Theory and Applications*, 37:265–293, 1988.

E. C. Kerrigan. *Robust Constraint Satisfaction: Invariant Sets and Predictive Control.* PhD thesis, University of Cambridge, 2000.

E. C. Kerrigan and J. M. Maciejowski. Feedback min-max model predictive control using a single linear program: robust stability and the explicit solution. *International Journal of Robust and Nonlinear Control*, 14(4):395–413, 2004.

H. Khalil. *Nonlinear Systems.* Prentice-Hall, 2 edition, 1996.

I. Kolmanovsky and E. G. Gilbert. Theory and conmputation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, 4:317–367, 1998.

M. V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32:1361–1379, 1996.

W. H. Kwon and A. E. Pearson. A modified quadratic cost problem and feedback stabilization of a linear system. *Automatica*, 22:838–842, 1977.

W. Langson, I. Chryssochoos, S. V. Rakovic, and D. Q. Mayne. Robust model predictive control using tubes. *Automatica*, 40:125–133, 2004.

J. H. Lee and Z. Yu. Worst-case formulations of model predictive control for systems with bounded parameters. *Automatica*, 33:763–781, 1997.

F. V. Lima and C. Georgakis. Design of output constraints for model-based non-square controllers using interval operability. *Journal of Process Control*, 18:610–620, 2008.

D. Limon. *Control Predictivo de sistemas no lineales con restricciones: estabilidad y robustez.* PhD thesis, Universidad de Sevilla, 2002.

D. Limon, T. Álamo, and E. F. Camacho. Stability analysis of systems with bounded additive uncertainties based on invariant sets: Stability and feasibility of MPC. In *Proceedings of the ACC*, 2002.

D. Limon, J. M. Bravo, T. Alamo, and E. F. Camacho. Robust MPC of constrained nolinear systems based on interval arithmetic. *IEE Control Theory and Applications*, 152:325–332, 2005.

D. Limon, T. Alamo, F. Salas, and E. F. Camacho. Input to state stability of min-max MPC controllers for nonlinear systems with bounded uncertainties. *Automatica*, 42: 629–645, 2006a.

D. Limon, T. Alamo, F. Salas, and E. F. Camacho. On the stability of MPC without terminal constraint. *IEEE Transactions on Automatic Control*, 42:832–836, 2006b.

D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho. MPC for tracking of piece-wise constant references for constrained linear systems. *Automatica*, 44:2382–2387, 2008a.

D. Limon, A. Ferramosca, I. Alvarado, T. Alamo, and E. F. Camacho. MPC for tracking of constrained nonlinear systems. In *Proceedings of 3rd International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control, NMPC 2008*, Pavia, Italia, September, 5-9 2008b.

D. Limon, T. Alamo, D. M. Raimondo, D. M. de la Peña, J. M. Bravo, A. Ferramosca, and E. F. Camacho. Input-to-state stability: an unifying framework for robust model predictive control. In L. Magni, D. M. Raimondo, and F. Allgöwer, editors, *International Workshop on Assessment and Future Direction of Nonlinear Model Predictive Control*, pages 1–26. Springer, 2009a.

D. Limon, A. Ferramosca, I. Alvarado, T. Alamo, and E. F. Camacho. MPC for tracking of constrained nonlinear systems. In L. Magni, D. M. Raimondo, and F. Allgöwer, editors, *International Workshop on Assessment and Future Direction of Nonlinear Model Predictive Control*, pages 315–323. Springer, 2009b.

D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho. Robust tube-based MPC for tracking of constrained linear systems with additive disturbances. *JPC*, 20:248–260, 2010a.

D. Limon, I. Alvarado, A. Ferramosca, T. Alamo, and E. F. Camacho. Enhanced robust NMPC based on nominal predictions. In *Proceedings of the 8th IFAC Symposium on Nonlinear Control Systems, NOLCOS 2010*, 2010b.

J. Liu, D. Muñoz de la Peña, B. J. Ohran, P. D. Christofides, and J. F. Davis. A two-tier Architecture for Networked Process Control. *Chem. Eng. Sci.*, 63(22):5394–5409, 2008.

J. Liu, D. Muñoz de la Peña, and P. D. Christofides. Distributed Model Predictive Control of Nonlinear Process Systems. *AIChE Journal*, 55(5):1171–1184, 2009.

D. E. Luenberger. *Linear and Nonlinear Programming.* Addison-Wesley, 1984.

L. Magni and R. Scattolini. Stabilizing decentralized model predictive control for nonlinear systems. *AUT*, 42(7):1231–1236, 2006.

L. Magni and R. Scattolini. On the solution of the tracking problem for non-linear systems with MPC. *Int. J. of Systems Science*, 36:8:477–484, 2005.

L. Magni and R. Scattolini. Tracking on non-square nonlinear continuous time systems with piecewise constant model predictive control. *Journal of Process Control*, 17: 631–640, 2007.

L. Magni, G. De Nicolao, L. Magnani, and R. Scattolini. A stabilizing model-based predictive control algorithm for nonlinear systems. *Automatica*, 37:1351–1362, 2001a.

L. Magni, G. De Nicolao, and R. Scattolini. Output feedback and tracking of nonlinear systems with model predictive control. *Automatica*, 37:1601–1607, 2001b.

L. Magni, H. Nijmeijer, and A. van der Shaft. A receding-horizon approach to the nonlinear H_∞ control problem. *Automatica*, 37:429–435, 2001c.

L. Magni, G. De Nicolao, and R. Scattolini. On robust tracking with non-linear model predictive control. 75:6:399 – 407, 2002.

D. Q. Mayne. Control of constrained dynamic systems. *European Journal of Control*, 7:87–99, 2001.

D. Q. Mayne and H. Michalska. Robust horizon control of nonlinear systems. *IEEE Transactions on Automatic Control*, 35:814–824, 1990.

D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.

D. Q. Mayne, M. M. Seron, and S. V. Rakovic. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41:219–224, 2005.

D. Q. Mayne, S. V. Rakovic, R. Findeisen, and F. Allgöwer. Robust output feedback model predictive control of constrained linear systems. *Automatica*, 42:1217–1222, 2006.

M. J. Messina, S. E. Tuna, and A. R. Teel. Discrete-time certainty equivalence output feedback: allowing discontinuous control laws including those from model predictive control. *Automatica*, 41:617–628, 2005.

H. Michalska and D. Q. Mayne. Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 38(11):1623–1633, 1993.

M. Morari, C. N. Jones, M. N. Zeilinger, and M. Baric. Multiparametric linear programming for control. In *Control Conference, 2008. CCC 2008. 27th Chinese*, pages 2–4. IEEE, 2008.

K. Muske. Steady-state target optimization in linear model predictive control. In *Proceedings of the ACC*, 1997.

K. Muske and J. B. Rawlings. Model predictive control with linear models. *AIChE Journal*, 39:262–287, 1993.

G. Pannocchia. Robust model predictive control with guaranteed setpoint tracking. *Journ. of Process Control*, 14:927–937, 2004.

G. Pannocchia and E. Kerrigan. Offset-free reciding horizon control of constrained linear systems. *AIChE Journal*, 51:3134–3146, 2005.

G. Pannocchia and J. B. Rawlings. Disturbance models for offset-free model-predictive control. *AIChE Journal*, 49:426–437, 2003.

G. Pannocchia, J. B. Rawlings, and S. J. Wright. Fast, large-scale model predictive control by partial enumeration. *AUT*, 43:852–860, 2007.

G. Pannocchia, S. J. Wright, B. T. Stewart, and J. B. Rawlings. Efficient cooperative distributed MPC using partial enumeration. In *Proceedings of ADCHEM*, pages 637–642, 2009.

S. J. Qin and T. A. Badgwell. A survey of industrial model predictive control technology. *Control Engineering Practice*, 11:733–764, 2003.

S. J. Qin and T. A. Badgwell. An overview of industrial model predictive control technology. In *Proceedings of the conference on Chemical Process Control*, 1997.

D. M. Raimondo, L. Magni, and R. Scattolini. Decentralized MPC of nonlinear systems: an input-to-state stability approach. *International Journal of Robust and Nonlinear Control, 17:1651-1667*, 2007a.

D. M. Raimondo, L. Magni, and R. Scattolini. Decentralized MPC of nonlinear systems: An input-to-state stability approach. *Int. J. Robust Nonlinear Control*, 17(17):1651–1667, 2007b.

D. M. Raimondo, D. Limon, M. Lazar, L. Magni, and E. Camacho. Min-max Model Predictive Control of Nonlinear Systems: A Unifying Overview on Stability. *European Journal of Control*, 15(1):5–21, 2009.

S. V. Rakovic, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne. Invariant approximations of the minimal robustly positively invariant sets. *IEEE Transactions on Automatic Control*, 50:406–410, 2005.

S. V. Rakovic, A. R. Teel, D. Q. Mayne, and A. Astolfi. Simple robust control invariant tubes for some classes of nonlinear discrete time systems. In *Proceedings of 45th IEEE Conference on Decision and Control, CDC 2006*, 2006.

C. V. Rao and J. B. Rawlings. Steady states and constraints in model predictive control. *AIChE Journal*, 45:1266–1278, 1999.

J. B. Rawligns and R. Amrit. Optimizing process economic performance using model predictive control. In L. Magni, D. M. Raimondo, and F. Allgöwer, editors, *International Workshop on Assessment and Future Direction of Nonlinear Model Predictive Control*, pages 315–323. Springer, 2009.

J. B. Rawlings and D. Q. Mayne. *Model Predictive Control: Theory and Design*. Nob-Hill Publishing, 1st edition, 2009.

J. B. Rawlings and K. R. Muske. Stability of constrained receding horizon control. *IEEE Transactions on Automatic Control*, 38:1512–1516, 1993.

J. B. Rawlings, D. Bonne, J. B. Jorgensen, A. N. Venkat, and S. B. Jorgensen. Unreacheable setpoints in model predictive control. *IEEE Transactions on Automatic Control*, 53:2209–2215, 2008.

R. T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

J. A. Rossiter, B. Kouvaritakis, and J. R. Gossner. Guaranteeing feasibility in constrained stable generalized predictive control. *IEEE Proc. Control theory Appl.*, 143: 463–469, 1996.

N. R. Sandell Jr., P. Varaiya, M. Athans, and M. Safonov. Survey of decentralized control methods for larger scale systems. *TAC*, 23(2):108–128, 1978.

P. O. M. Scokaert and D. Q. Mayne. Min-max feedback model predictive control for constrained linear systems. *IEEE Transactions on Automatic Control*, 43(8):1136–1142, 1998.

P. O. M. Scokaert, J. B. Rawlings, and E. S. Meadows. Discrete-time stability with perturbations: Application to model predictive control. *Automatica*, 33(3):463–470, 1997.

P. O. M. Scokaert, D. Q. Mayne, and J. B. Rawlings. Suboptimal model predictive control (feasibility implies stability). *IEEE Transactions on Automatic Control*, 44 (3):648–654, 1999.

B. T. Stewart, A. N. Venkat, J. B. Rawlings, S. J. Wright, and G. Pannocchia. Cooperative distributed model predictive control. *Systems & Control Letters*, 59:460–469, 2010.

M. Sznaier and M. J. Damborg. Suboptimal control of linear systems with state and control inequality constraints. In *Proceedings of the CDC*, 1987.

P. Tatjewski. Advanced control and on-line process optimization in multilayer structures. *Annual Reviews in Control*, 32:71–85, 2008.

A. N. Venkat. *Distributed Model Predictive Control: Theory and Applications.* PhD thesis, Univeristy of Wisconsin - Madison, October 2006.

A. N. Venkat, J. B. Rawlings, and S. J. Wright. Distributed Model Predictive Control of Large-Scale Systems. *Lecture Notes in Control and Information Sciences*, pages 591–605, 2007.

V. Vesely, Z. Kralova, L. Harsanyi, and K. S. Hindi. Modified feasible method for hierarchical steady-state control of complex sytems. *IEE Proc. Control Theory Appl.*, 145, 1998.

D. R. Vinson and C. Georgakis. A new measure of process output controllability. *Journal of Process Control*, 10:185–194, 2000.

Z. Wan and M. Kothare. An efficient off-line formulation of robust model predictive control using linear matrix inequalities. *Automatica*, 39:837–846, 2003a.

Z. Wan and M. V. Kothare. Efficient scheduled stabilizing model predictive control for constrained non linear systems. *Int. J. of Robust and Nonlinear Control*, 13:331–346, 2003b.

Y. Wang. *Robust Model Predictive Control.* PhD thesis, Univ. of Wisconsin-Madison., 2002.

L. Würth, J. B. Rawlings, and W. Marquardt. Economic dynamic real-time optimization and nonlinear model predictive control on infinite horizons. In *Proceedings of the International Symposium on Advanced Control of Chemical Process*, Istanbul, Turkey, 2009.

A. C. Zanin, M. T. de Gouvêa, and D. Odloak. Integrating real time optimization into the model predictive controller of the fcc system. *Control Engineering Practice*, 10: 819–831, 2002.

M. N. Zeilinger, C. N. Jones, and M. Morari. Real-time suboptimal model predictive control using a combination of explicit MPC and online optimization. *Automatic Control, IEEE Transactions on*, (99):1–1, 2008.